ANALYSIS OF PARTIAL DIFFERENTIAL ALGEBRAIC EQUATIONS

by

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Abstract

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In this thesis, we analyse infinite dimensional differential algebraic equations (DAEs). First, we extend the notion of *index* to partial DAEs. Three different type of indices (modal, perturbation and algebraic) are defined and compared with each other. The comparison with finite dimensional DAEs is also done. It is shown that infinite dimensional DAEs exhibit richer behavior than finite dimensional DAEs, since the former may have solutions (and indices) which depend not only on the forcing functions, data (initial and boundary conditions), but also on the region of interest and method of approximation. In particular, the method of lines (MOL) with finite difference and finite element approximation is analysed. Then, we analyse the traveling wave solutions in both linear and nonlinear infinite dimensional systems. The solutions of dissipative systems of conservation laws in gas dynamics and magnetohydrodynamics (MHD) naturally lead to DAEs when one looks for a special solution in the form of a traveling wave between the *left* and *right* equilibria. The structure of the DAE (semi-explicit or conservative) depends on the dissipative mechanism involved. Next, we analyse the singularity induced bifurcation in MHD, when an equilibrium is placed at the singularity of a MHD DAE. It is shown that one may be able to integrate through the singularity to connect two equilibria lying on the opposite sides of that singularity. In some cases we reach and leave the equilibrium at the singularity in finite time. Our analysis is illustrated by many numerical examples. We also present a few related research topics for further research.

To Alina, Joanna and Adam

Biography

The author was born in Nowa Deba, Poland, on December 10, 1957. He graduated from Tarnobrzeg High School in 1976 and enrolled in the Technical University of Warsaw, Poland, first as an undergraduate and then as a graduate student. He received a Ph.D. in electrical engineering (control theory) from that university in 1984. From 1984 till 1990 he worked as an assistant professor in control engineering at the Technical University of Opole, Poland. This period includes an involuntary "sabbatical" leave of a one year of military service. In 1990 he received a Humboldt Research Fellowship from the Humboldt Foundation in Germany and spent 2 years doing research at Bochum University in Germany. Since 1992 he has lived in the USA. He received a Ph.D. in applied mathematics from North Carolina State University in 1996. He is an author or coathour of about 40 journal and conference papers.

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1 Introduction to Differential Algebraic Equations

The differential algebraic equation, or DAE, is an equation of the form

$$F(u', u, t) = 0 \tag{1}$$

where the partial derivative $\frac{\partial F}{\partial u'}$ is identically singular. Here, $F : \Omega \subseteq \mathbf{R}^{2m+1} \rightarrow \mathbf{R}^s$, where Ω is an open connected set. Many physical systems are initially modelled by DAEs. The most important areas where DAEs occur are the constrained variational problems, constrained path control problems in mechanical engineering (e.g. robotics), modeling of electrical circuits, and the discretization of partial differential equations (PDEs) by the method of lines or the method of moving grids. Many problems in the real world are initially modeled as (1). Then, some problems, after additional manipulations on the equations involved (reduction of variables), may be transformed into an ordinary differential equation, or ODE. However, there is a large class of problems for which such reduction is not possible. Examples may be found in [18],[50],[51]. Those problems must be treated as DAEs. Additionally, even if the original problem does not seem to be modeled by a DAE, it may eventually end up as a DAE problem. For example, this is often the case if we make approximations, or consider a specific type of solution to the problem. An example of this kind is considered in this thesis, where we look for a specific solution, namely the traveling wave solution in the systems of conservation laws.

Historically, the very first attempts to solve DAEs involved reduction of DAEs to ODEs by tedious manipulations. Then ODEs were solved by numerical integrators design exclusively for ODEs [17], [28]. Next, different approach has been developed: to solve DAEs directly. Thus the concept of solvability of DAEs had to be formulated and analyzed in some detail. This concept is essentially based on an observation that DAEs are differential equations defined on submanifolds of $\mathbf{R}^s$. The requirement is that (1) is solvable in the sense that it has a family of solutions which are uniquely determined by the values at a given $t = t_0$. These solutions have to form a manifold of integral curves, the solution manifold. The formal definition of the solvability of DAE is as follows.

Definition 1.1 *[10] Let I be an open subinterval of $\mathbf{R}$, Ω a connected open subset of $\mathbf{R}^{2s+1}$, and F a differentiable function from Ω to $\mathbf{R}^s$. The DAE (1) is solvable on I in Ω if there exists an r dimensional family of solutions $\phi(t,c)$ defined on a connected open set $I \times \tilde{\Omega}$, $\tilde{\Omega} \subseteq \mathbf{R}^r$, such that:*

(i) $\phi(t,c)$ is defined on all of I for each $c \in \tilde{\Omega}$;

(ii) $(\phi_t(t,c), \phi(t,c), t) \in \Omega$ for $(t,c) \in I \times \tilde{\Omega}$;

(iii) if $\psi(t)$ is any solution with $(\psi'(t), \psi(t), t) \in \Omega$, then $\psi(t) = \phi(t, c)$ for some $c \in \tilde{\Omega}$; and

(iv) the graph of ϕ as a function of (t, c) is an $(r+1)-$dimensional manifold.

Several issues follow from this definition. First of all, the definition does not impose condition of how smooth the solution of a solvable DAE should be. Second, the DAE has exactly one solution in the case $r = 0$.

The following example illustrates how DAE may differ from ODEs.

Example 1.1 *[14] The solution of the semi-explicit DAE*

$$u_1' = u_2 + \delta_1(t) \tag{2a}$$

$$0 = \beta(t)u_1 + \delta_2(t) \tag{2b}$$

is $u_1 = -\beta^{-1}(t)\delta_2(t)$, $u_2 = -[\beta^{-1}(t)\delta_2(t)]' - \delta_1(t)$ which involves derivatives of both β and δ. Also, not all initial data $(u_1(0),\ u_2(0))$ will be consistent for (2) if we restrict ourselves to continously differentiable solutions.

1.1 Basic types of DAEs

Equation (1) is the most general form of a DAE. It is called *a fully-implicit* DAE. There are also several special forms of (1) and a particular method of solving and analyzing of a DAE may depend on such forms. One of the most commonly used forms of DAEs is *the semi-explicit* form, The semi-explicit DAE is defined as

$$u_1' = f_1(u_1, u_2, t) \tag{3a}$$

$$0 = f_2(u_1, u_2, t) \tag{3b}$$

where (3b), a purely algebraic equation represents a constraint. A DAE may also have what is called *a hidden constraint* which can be derived only after some additional manipulation on equations.

We can use the semi-explicit DAE to show its relationship to an ODE. Let us consider a solvable DAE (3) and differentiate (3b) with respect to t, as follows

$$0 = \frac{\partial f_2}{\partial u_1}u_1' + \frac{\partial f_2}{\partial u_2}u_2' + \frac{\partial f_2}{\partial t}. \tag{4}$$

If $\frac{\partial f_2}{\partial u_2}$ is nonsingular, then from (4) we obtain

$$u_2' = -\left(\frac{\partial f_2}{\partial u_2}\right)^{-1}\left(\frac{\partial f_2}{\partial u_1}u_1' + \frac{\partial f_2}{\partial t}\right) \tag{5a}$$

$$= -\left(\frac{\partial f_2}{\partial u_2}\right)^{-1}\left(\frac{\partial f_2}{\partial u_1}f_1(u_1, u_2, t) + \frac{\partial f_2}{\partial t}\right) \tag{5b}$$

Thus (3a) together with (5b) gives an ODE. If we assume that this ODE is solvable and a solution of the ODE satisfies (3b) at $t = t_0$, then this solution must also satisfy the DAE by the solvability assumptions.

Another special DAE is a DAE in Hessenberg form. The Hessenberg forms of size two and three are the most common. The nonlinear Hessenberg form of size two is

$$u_1' = g_1(u_1, u_2, t) \tag{6a}$$

$$0 = g_2(u_1, t) \tag{6b}$$

with $\frac{\partial g_2}{\partial u_1}\frac{\partial g_1}{\partial u_2}$ nonsingular.

The Hessenberg form of size three is

$$u_1' = g_1(u_1, u_2, u_3, t) \tag{7a}$$

$$u_2' = g_2(u_1, u_2, t) \tag{7b}$$

$$0 = g_3(u_2, t) \tag{7c}$$

where $\frac{\partial g_3}{\partial u_2}\frac{\partial g_2}{\partial u_1}\frac{\partial g_1}{\partial u_3}$ is nonsingular. Note that, in general, none of these matrices of partial derivatives are square much less are they invertible. It is only the product that needs to be square and nonsingular.

The Hessenberg form DAEs are important equations of the mechanics and variational problems. Some beam deflection problems can actually be in Hessenberg form of size four [31]. The Hessenberg form DAEs were also important from the point of view of the development of numerical packages for solving DAEs. The programs like RADAU5 [38] can be used for solving DAEs in Hessenberg form.

The fourth special form of DAE is *a conservative* DAE which took its name from the fact that such a DAE is usually found when analyzing system of conservation laws. The general structure of a DAE in conservation form is

$$[h(u(t))]' = f(t) \tag{8}$$

or

$$h_u(u(t))u'(t) = f(t) \tag{9}$$

DAEs in conservation form are considered in Section 5 in this thesis. It is possible to transform (8) to a semi-explicit form (3), but this usually increases the *index* of a DAE. The *index* is one of the most important notions in both linear and nonlinear DAEs. Different types of indices are summarized below.

1.2 Index of DAE

Let us for a moment go back to a semi-explicit DAE (3). If $\frac{\partial g}{\partial u_2}$ is nonsingular then we are able to transform (3) to an ODE with just one differentiation of the constraint (3b). We say that a semi-explicit DAE with such property has *index* equal 1. If $\frac{\partial g}{\partial u_2}$ is singular, suppose that with algebraic manipulation and coordinate changes we can rewrite (4) in the form of (3b) but different u_1 and u_2. We differentiate that new constraint equation and see if it can be transformed to an ODE. If so, the original DAE will have index equal 2. If not, we proceed in the same manner until we obtain an ODE. Each differentiation of the transformed constraint equation increases the index by 1. Thus we have the following definition.

Definition 1.2 *The minimum number of times that all or part of (1) is differentiated with respect to t in order to determine u' as a continuous function of u and t, is the index of the DAE (1).*

Example 1.2 *[1] Consider the following semi-explicit DAE*

$$
\begin{aligned}
u_1' + u_3 &= f_1 &\quad (10a)\\
u_2' + u_1 &= f_2 &\quad (10b)\\
u_2 &= f_3 &\quad (10c)
\end{aligned}
$$

If we differentiate (10a), (10b) and (10c), once, twice and three times, respectively, then we have

$$
\begin{aligned}
u_1'' + u_3' &= f_1' &\quad (11a)\\
u_2''' + u_1'' &= f_2'' &\quad (11b)\\
u_2''' &= f_3''' &\quad (11c)
\end{aligned}
$$

Then, adding (11a), (11c), and subtracting (11b), we obtain

$$u_3' = f_1' - f_2'' + f_3''' \tag{12}$$

Two differentiations of (10c) and one differentiation of (10b) yield

$$u_1' = f_2' - f_3'' \tag{13}$$

One differentiation of (10c) gives

$$u_2' = f_3' \tag{14}$$

Since we need three differentiation to obtain u', the index is 3.

The differentiations described above lead to what is known as the *derivative array equations* which plays important role in the analysis of DAEs as well as the development of numerical algorithms for DAEs. The derivative array equations are

$$\begin{bmatrix} F(u', u, t) \\ F_t(u', u, t) + F_u(u', u, t)u' + F_{u'}(u', u, t)u'' \\ \vdots \\ \dfrac{d^k}{dt^k} F(u', u, t) \end{bmatrix} = G_k(u', u, t, w) = 0 \tag{15}$$

where $w = [u^{(2)}, \ldots, u^{(k+1)}]$. We frequently drop the k subscript on G to simplify our notation.

As the Definition 1.2 states, the index of a DAE (1) is k if $G_k = 0$ uniquely determines u' given consistent u and t. In this way we have what is called the *global* or *differentiation* index, frequently denoted by ν_d.

The *perturbation* index, ν_p introduced in [38] is defined in the folowing way.

Definition 1.3 *The DAE (1) has perturbation index ν_p along a solution u on the interval $\mathcal{I} = [0, T]$ if ν_p is the smallest integer such that if*

$$F(\hat{u}', \hat{u}, t) = \delta(t) \tag{16}$$

for sufficiently smooth δ, then there is an estimate

$$||\hat{u}(t) - u(t)|| \leq C \left(||\hat{u}(0) - u(0)|| + ||\delta||^t_{\nu_p - 1} \right) \tag{17}$$

in the $||.||_{\nu_p-1}$ norm. C is a constant that depends on F, T and u.

It has been shown [9] that both the differentiation and perturbation indices can differ significantly for a fully-implicit nonlinear DAEs (1).

The third type of index, the *uniform differentiation index* is defined in the following way. Let Ω^e (an extended neighborhood) be an open set in (t, u, u', w) space, and G_k be as in (15).

Definition 1.4 *The uniform differentiation index, ν_{UD} of the DAE (1) on Ω^e is the smallest integer ν, if it exists, such that the following four conditions hold on Ω^e:*

(A1) Sufficient smoothness of $G = G_{\nu+1} = 0$.

(A2) Consistency of $G = 0$ as an algebraic equation.

(A3) $\overline{J}_\nu = [G_{u'}\ G_w]$ is 1-full and has constant rank.

(A4) $J_\nu = [G_{u'}\ G_w\ G_u]$ has full row rank independent of (t, u, u', w).

It is known [7] that if (A1), (A2) and (A4) hold for one value of ν, then they also hold for lower values of ν. This is not the case with (A3). Other properties and comparisons between different indices of DAEs can be found in [9],[10],[38]. The indices presented above will be analysed in more detail in the context of infinite dimensional DAEs in Section 3 in this thesis.

2 Why Partial Differential Algebraic Equations?

While considerable progress has been achieved in the analysis of (1), or its various forms, it appears that not much work has been done to link (1) to infinite dimensional systems described by partial differential equations or PDEs. It turns out that many PDE problems, due to their complexity, may be described by systems which naturally lead to (1). Also, one may use different approximating techniques, such as the method of lines (MOL), modal expansion, or traveling wave solution to obtain (1) even when the original problem does not seem to be in any *singular*, *implicit*, or *degenerate* form. The goal of this thesis is to show how to use the technique of DAEs in infinite dimensional systems. A variety of problems will be considered stretching from purely theoretical issues in partial DAEs (PDAEs) to interesting applications in boundary control problems, systems of conservation laws in gas dynamics and magnetohydrodynamics.

Why is it important to link (1) to infinite dimensional problems? There exists several reasons for this. First, even when a solution of any particular PDE problem is known it is usually useful to look at the result from a different perspective. Therefore it may be interesting to look at the PDE problem and its solution from the DAE perspective and see what one can learn from that. An observation of this kind is given in Section 5, where after presenting the travelling wave MHD DAE in Section 4, we discuss several implications of this interesting DAE for those interested in developing numerical integrators for general unstructured DAEs (see comments in Subsection 6.3). This is connected with the second reason, namely, looking at a PDE problem from the DAE perspective may help to identify new ideas and research topics in DAEs (e.g. research on DAEs with Jacobians with variable null spaces, or null spaces dependent on x and x'; see Subsection 5.5). Third, by showing the links between some theoretical notions of DAEs and properties of infinite dimensional systems one is often able to attach a physical meaning to those notions. For example, changes in the index of a DAE are closely related to the *sonic* points in the travelling wave DAE for systems of conservation laws (Subsection 5.5). Fourth, it is believed that there are no results reported linking the DAE theory to that of travelling waves (and therefore shock solutions) in gas dynamics and magnetohydrodynamics and in this respect results presented in Section 5 are new. Fifth, what looks like a motivational aspect, it is believed that showing several possible applications of (1) in PDE problems will stress the importance of research work on DAEs which exhibit much richer behavior than ordinary differential equations or ODEs. Sixth, one can find many mathematical models of engineering systems (e.g. in mechanics and chemical process control) that are written in PDAE forms. They are usually converted to DAEs, but little consideration has been given to examine this approximation process.

Given the above motivation we set the goal of this thesis as to apply the DAE perspective to several infinite dimensional problems, identify possible relations between known results in DAE theory and PDE problems, see how the results differ from what we know when (1) is applied in lumped parameter systems, interpret the results in terms of DAE concept (e.g. its index), and finally give a *feedback* as to what is needed to be researched on in DAEs in order to solve some application problems.

We will usually use the abbreviation PDAE where the infinite dimensional system (distributed parameter system) is described by a system of linear PDEs with singular Jacobian with respect to $\dot{x}$ (Section 3, and Subsections 4.1, 4.2 and 4.4). In other cases, like travelling wave solutions in gas dynamics (GD) or magnetohydrodynamics (MHD), we prefer to use the term travelling wave GD (or MHD) DAE. Similarly, the equation describing the boundary control problem (BCP) considered in Section 4.3 is referred to as the BCP DAE.

3 The Index of an Infinite Dimensional Implicit System

3.1 General Differential Algebraic Equation

Infinite dimensional systems are more complex than finite dimensional ones, because of the greater number of ways that functions can enter the equations. Rather than trying to give a completely correct general definition we shall first describe what is needed to be taken into consideration when defining an index of an infinite dimensional partial differential algebraic equation. Then more concrete definitions of various indices will follow.

The linear partial differential algebraic equation or PDAE we consider here has the general form

$$Au_t + Bu_{xx} + Cu + Du_x = f(x,t) \tag{18}$$

where $u \equiv u(x,t) \in \mathbf{R}^n$, $x \in (0,L)$, $t > 0$, $L > 0$, A, B, C, $D \in \mathbf{R}^{n\times n}$, $detA = 0$ and $f(x,t) \in \mathbf{R}^n$.

We will also usually assume the following boundary and initial conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = g(x) \tag{19}$$

where f and g are smooth enough and are consistent with the boundary and initial conditions.

System (18),(19) can sometimes be regarded as a limit case of a singularly perturbed parabolic system with $A = A(\epsilon)$ where $\epsilon \in \mathbf{R}$ and $detA(\epsilon) = 0$ if $\epsilon = \epsilon_0$. Systems with A singular also arise directly in several areas.

The discontiuous and impulsive solutions that may exist in PDAEs raise a natural question of what meaning can be given to the derivatives of u, f and g. Obviously, in the case of impulsive or discontinuous solutions, u_t, u_{xx}, and other derivatives, cannot be "functions" in the usual sense.

The results presented in this chapter concern the 3 major indexes which can be associated with linear or nonlinear PDAEs. The definitions of 2 of those indexes (perturbation and modal) include either derivative terms (perturbation index) or involve linear finite dimensional DAEs resulting from the Fourier analysis of the infinite dimensional DAEs (modal index). We shall consider these indexes in the context of smooth solutions of PDAEs only. We assume that the inputs and data can be differentiated sufficiently many times, so that it makes sense to talk about the special sup-type norms defined in (26)-(28) later on. Therefore, in chapter 3 we define, analyze and compare various indexes in the case of PDAEs with smooth solutions.

If for some reason one is not satisfied with the requirements of the smooth solutions it would be natural to turn to the generalized functions, or distributions. Let us mention the basic idea that stands behind the generalized solutions.

Given a function u, we define a linear mapping

$$\phi \rightarrow \int_{\Omega} u(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}) \tag{20}$$

where $\mathbf{x} \in \Omega$ and $\phi \in C^{\infty}(\Omega)$ is a test function which vanishes on and outside Γ, the boundary of Ω [27].

Then, the derivative $\partial u/\partial x_i$, $x_i \in \Omega$ is defined as the mapping

$$\phi \rightarrow -\int_{\Omega} u(\mathbf{x})\frac{\partial \phi(\mathbf{x})}{\partial x_i} \tag{21}$$

Clearly, this definition requires no differentiablity of u in the usual sense. The only differentiability requirement is on ϕ. If Ω has a smooth boundary Γ, and if $u \in C^1(\Omega \setminus \Gamma)$, $u = 0$ in the exterior of Ω, then the right hand side of the last expression becomes

$$\int_{\Omega} \frac{\partial u(\mathbf{x})}{\partial x_i}\phi(\mathbf{x})d\mathbf{x} - \int_{\Gamma} u(\mathbf{x})\phi(\mathbf{x})n_i dS \tag{22}$$

where n_i is the unit outward vector at the boundary Γ. That is, the distributional derivative of u involves a term corresponding to the ordinary derivative and another term involving a distribution supported on Γ. The latter term results from a jump of u accross Γ. Then, this term will accomodate the distributional terms present in solutions of (18) at the three sides of $\Omega = [0, L] \times t$, $t \geq 0$. Note that turning to the generalized solutions of PDAEs would require appropriate changes in the definitions of indexes, since one cannot use the same approach for both generalized and smooth solutions. In particular, our sup-type norms (26)-(28) are not valid for generalized solutions.

Note that impulsive and discontinuous solutions are sometimes allowed in physical systems, for example in electrical circuits. On the other hand, in many applied areas only consistent initial conditions and smooth solutions are considered. Therefore, in the xi“”applied” part of this thesis considered in Chapter 5 we consider only smooth solutions. The traveling waves considered in Chapter 5 are smooth solutions. It is true that the limit of the smooth traveling wave solutions can be discontinuous, but we do not directly consider this limiting solution. Also, in Chapter 4 we show that smooth solutions of the MOL approximations will sometimes be approximations of distributional solutions of PDAEs.

Section 4.4 in chapter 4 deals with another type of smooth traveling wave solutions of PDAEs and is included in this thesis to illustrate a different nature of the wave solutions in infinite dimensional problems. This approach is not used in the analysis in chapter 5.

To make our presentation as clear as possible, we explicitely state at the beginning of each chapter what type of solutions are we dealing with.

Before giving a detailed analysis of what we mean by the index of (18),(19), note that (18),(19) can be regarded as a particular case of a more general DAE, or GDAE. Therefore, in order to generalize the concept of index from the finite dimensional DAEs, we take the following approach.

Let D_i be various differential operators and the L_i be integral operators. For the subset Ω_i of $\mathbf{R}^{n_i}$ we let $\Omega = \Pi_{i=1}^{r}\Omega_i$ and Σ be a set of functions u defined on Ω. Let the u be vector valued with the finite dimensional ranges. A set $\mathcal{F}$ of functions f defined on a subset $\Omega_{\mathcal{F}}$ of Ω is called a set of forcing functions or inputs, while the $u's$ are called the states. In general, the forcing functions need not have the same domain as the states. This happens in various boundary control problems or specific outputs (e.g. [11]).

The third set $\mathcal{D}$ of functions g, defined on a subset $\Omega_{\mathcal{D}}$ of Ω is called the data. We assume that there is finite number of linear operators D_i with right inverses L_i. D_i and L_i may be defined on data, inputs, or states. Let P denote a polynomial in various L_i, D_i, We assume that it makes sense to add, or multiply the given operators in P.

A generalized DAE (or GDAE) consists of two relationships, namely

$$F(P_1u, P_2u, ..., P_ru, P_{r+1}f, ..., P_mf) \quad = \quad 0 \tag{23a}$$

$$B(P_{m+1}u, ..., P_su, P_{s+1}g, ..., P_Ng) \quad = \quad 0 \tag{23b}$$

The relationship (23a) is the actual DAE. The second relationship (23b) describes how the data and the state are to be related.

Data are those conditions on solutions of (23a) that we would like to hold. The data is the generalization of the idea of an initial condition for a finite dimensional DAE. Those conditions which must hold are incorporated into the definitions of the different sets Ω, $\mathcal{F}$, $\mathcal{D}$. Examples might include smoothness assumptions or boundary conditions for a PDE. This distinction of which are and are not data must be made by the user and has important consequences for the interpretation of the solution.

There are different ways to proceed at this point. In some formulations the operators and functions are treated as purely algebraic objects. This is the case with the differential algebra approach. In the next section we give a specific example where we define a purely algebraic index. This approach can be very useful and provides considerable insight [21],[22]. However, if we are interested in numerical analysis, we need to take a different approach. In numerical settings the various sets and spaces are equipped with topologies and it makes sense to talk of continuous operators. As we will show in this section properties of the numerical solution of a constrained PDE can be dependent not only on the algebraic properties of the operator but also on quantities such as the domain which do not play an obvious part in the differential geometric theory. For concreteness we will assume that the topologies are given by some type of norm. This terminology is adapted from the finite dimensional cases but there are certain differences to be pointed out later. For a given f, u is a solution of (23a) if it satisfies the equation (23b) in the specified sense. Typically this is pointwise or almost everywhere. A data g is *consistent* (for a given f) if there is a solution $u \in \Sigma$ which satisfies the data, that is (23a) holds for this f, g. Consistent with the finite dimensional case we say that the GDAE is *solvable* if for every forcing function f there are solutions of the GDAE for some data. These solutions are uniquely determined by consistent data. One can restrict the concept of solvability to being a local concept. This is often done with nonlinear problems.

The solutions u depend on the data and the forcing functions. Our definition of the index generalizes that of the perturbation index [38]. We say that the *infinite pointwise perturbation index* at $\hat{u}$, which satisfies (23) with $\hat{f}$, $\hat{g}$, is ν_p^∞ if a solution u of (23) with f and consistent g satisfies

$$||u - \hat{u}|| \leq \sum_{j=0}^{r_1} M_j ||P_j(f - \hat{f})|| + \sum_{j=0}^{r_2} B_j ||Q_j(g - \hat{g})|| \tag{24}$$

where B_j, M_j are nonzero constants, P_j, Q_j are fixed polynomials in the D_i, and ν_p^∞ is one more than the largest *combined power* of the D_i that occurs. Combined power is the sum of the powers of the the D_i in any one product in a P_j. $||.||$ may depend on j, P_j, Q_j. The *(maximum) perturbation index* at $\hat{u}$ is the maximum of ν_p^∞ in a u neighborhood of $\hat{u}$.

Note that (24) has greater complexity than the formula in the finite dimensional case. (24) includes not only derivatives of f but also of the data g. Also, the index depends on the particular choice of the basic operators D_i and polynomials P_j, Q_j. One of the advantages of implicit models is that they make it easy to change ones mind about what is input and what is output and what is data. However, the index is dependent on making a particular choice.

3.2 Perturbation Index

We now turn to carefully examining (18) with the boundary and initial conditions (19). Restricting ourselves to this specific class of linear time invariant PDAE will allow us to make several observations. In particular, we will show how our approach differs from the algebraic approach. We will frequently consider the special case of (18) with $C = 0$, $D = 0$. This system is explored in [18], [19] where the eigenfunctions for the operator $\partial^2/\partial x^2$ were used. Here, we apply an orthogonal basis. At this point we shall keep our calculations somewhat formal, knowing that one need to consider the question of convergence. Let $g_j(x) = \sin(n\pi x/L)/\sqrt{L}$, $\lambda_n = -(n\pi/L)^2$. we can then consider the series

$$u(x,t) = \sum_{j=1}^{\infty} g_j(x) u_j(t), \quad f(x,t) = \sum_{j=1}^{\infty} g_j(x) f_j(t), \quad u_0(x) = \sum_{j=1}^{\infty} u_{0j} g_j(x) \tag{25}$$

In general, $c_j(t)$ is the jth coefficient of $c(x,t)$ with respect to $g_j(x)$. We need to define some norms in order to be precise about the perturbation index. Let $||\cdot||$ be the usual Euclidian norm on $\mathbf{R}^n$. For a function $c(x,t)$ we define $||c||_\infty$ to be

$$||c||_\infty = \max_{0\leq t\leq T}\left(\int_0^L ||c(x,t)||^2 dx\right)^{1/2} = \max_{0\leq t\leq T}||c(x,t)||_2 \tag{26}$$

where $||.||_2$ is the usual L_2 norm in the x variable. Equivalently,

$$||c||_\infty = \max_{0\leq t\leq T}\left(\sum_{j=1}^{\infty}||c_j(t)||^2\right)^{1/2} \tag{27}$$

Finally, we define

$$||c||_{(p,q)} = \sum_{i=0}^{p}\sum_{k=0}^{q}\max_{0\leq t\leq T}\left(\sum_{j=1}^{\infty}||c_j^{(i)}(t)||^2(\frac{j\pi}{L})^{2k}\right)^{1/2} = \sum_{i=0}^{p}\sum_{k=0}^{q}||\frac{\partial^{i+k}c(x,t)}{\partial t^i\partial x^k}||_\infty \tag{28}$$

Note that $||c||_{(p,q)}$ being finite implies that one can take p t-derivatives and q x-derivatives of c term by term.

The estimate (24) then takes the form

$$||u-\hat{u}|| \leq C_1||f-\hat{f}||_{(p_1,q_1)} + C_2||g-\hat{g}||_{(p_2,q_2)} \tag{29}$$

We take the perturbation index ν_p^∞ of (18),(19) to be

$$\nu_p^\infty = 1+\min\{\max\{p_1+q_1, p_2+q_2\} : \text{ (25) holds for } (p_1,q_1,p_2,q_2)\} \tag{30}$$

In this definition of the perturbation index we have taken $D_1 = \partial/\partial t$, $P_j(z) = z^j$, $D_2 = \partial/\partial x$, $Q_i(z) = z^i$. This is the most natural choice in terms of the usual concept of smoothness. However, since the partials always occur an even number of times we could have taken $D_2 = \partial^2/\partial x^2$. This would have altered the value of the index for some problems.

3.3 Algebraic Index

The algebraic approach to time invariant PDAEs has the characteristic that it tends to be independent to some extend of the data.

Suppose that $r(s,z)$ is a fraction of two real polynomials in the real variables s, z. We say that r is *s-proper* if $lim_{|s|\to\infty}r(s,z) = 0$ for almost all z. *z-proper* is defined the same way. We say that r is *proper* if is both $s-$ and *z-proper*. A matrix has a given properness property if every one of its entries has that property.

We call (18) *regular* if $det(sA+z^2B+C+Dz) \not\equiv 0$. Regularity is nessary for solvability. To see this suppose that we do not have regularity. Then there is a matrix polynomial

$D(s,z)$ such that $(sA+z^2B+C+Dz)E(s,z)=0$. Let b be any function of (x,t) for which enough derivatives of b satisfy the boundary conditions. Then $E(\frac{d}{dt},\frac{d^2}{dx^2})b$ is a solution of the associated homogeneous equation (18),(19).

Assuming we have regularity, let

$$R(s,z)=(sA+z^2B+C+Dz)^{-1} \tag{31}$$

We define the *algebraic t-index* ν_t of (18) to be the smallest integer n_1 such that $s^{-n_1}R(s,z)$ is *s-proper*. The *algebraic x-index* ν_x of (18) is the smallest n_2 such that $z^{-n_2}R(s,z)$ is *z-proper*. We want the algebraic index to capture, among other things, the highest degree of smoothness required of f. This leads to the following definition. let $R_{i,j}$ be the i,j entry of R. Then the *algebraic index* of (18) is

$$\nu_A^{\infty}=\max_{i,j}\{\min_{n_1,n_2\geq 0}\{n_1+n_2 : z^{-n_1}s^{-n_2}R_{i,j}(s,z) \text{ is proper}\}\} \tag{32}$$

Before considering (32) in some detail we note that regularity is no longer sufficient in general to guarantee solvability.

Proposition 3.1 *Suppose in (18) that $C=0$, $D=0$. Suppose that the boundary conditions in (19) are replaced by linear homogeneous boundary conditions for which 0 is an eigenvalue of $\frac{\partial^2}{\partial x^2}$. Then (18) is not solvable for any A,B if A is singular.*

Proof. Let $\gamma(x)$ be a scalar eigenfunction for 0 for $\frac{\partial^2}{\partial x^2}$. Thus γ satisfies the boundary conditions. Let $Av=0$ and $\psi(t)$ be any smooth scalar function. Then $u=\gamma(x)\psi(t)v$ is a solution of the homogeneous PDE which satisfies the boundary conditions. The initial condition u_0 determines only $\psi(0)$ and not $\psi(t)$. □

A simple example of the above phenomena is gotten by taking the boundary conditions to be $u_x(0,t)=u_x(L,t)=0$. Another example is Example 3.1.

3.4 Modal Index

We consider now the case where $D=0$. Under the assumption that u and f are smooth enough in t and x we may substitute the series (25) for u, f into the PDAE (18) and get the *modal* DAEs

$$Au_j'(t)+(\lambda_jB+C)u_j(t) = f_j(t), \quad j\geq 1 \tag{33a}$$

$$u_j(0) = u_{0j} \tag{33b}$$

The solutions of the *modal* DAEs are determined by the parametrized family of matrix pencils

$$\mathcal{P}_{\lambda_j} = \{A, \lambda_j B + C\} \tag{34}$$

It is important to note the different roles played by the different components of (34). The matrices A, B, C come from the PDAE (18). However, the λ_j are discrete numbers depending on L, j. In general, there are several ways to interpret (18),(19), depending on what is chosen as the state and what is free. Here, we consider (18), (19) in the usual situation where f is free and u is the state.

We take $\Omega = \Omega_{\mathcal{F}} = [0,1] \times [0,T]$, $\Omega_{\mathcal{D}} = [0,1] \times \{0\}$. We include in the definitions of Σ, $\mathcal{F}$ that the boundary conditions in (19) are defined.

We shall say that (18) is *modal solvable* if (33) is solvable for every j. This means that we incorporate some smoothness assumptions into our spaces Ω. Let ν_j be the index of (33a) if (34) is a regular pencil. Then the *modal index* is

$$\nu_M^\infty = \max_j \{\nu_j\} \tag{35}$$

Proposition 3.2 *Suppose that $\mathcal{P}_\alpha$ is a regular pencil for every $\alpha < 0$, then (18) is a modal solvable PDAE for any L. However, if there exists a number $\hat{\alpha} < 0$ for which $\mathcal{P}_{\hat{\alpha}}$ is not regular and a number $\bar{\alpha}$ for which $\mathcal{P}_{\bar{\alpha}}$ is regular, then (18) is modal solvable for all L except for a countable number of L where it is not solvable.*

Proof. The first statement is clear. So suppose that $\mathcal{P}_{\bar{\alpha}}$ is regular. Note that $det(\lambda A + \alpha B + C)$ is a polynomial in two variables. Thus (34) is regular for α near $\bar{\alpha}$. On the other hand (34) is not regular if $L = n\pi(-\hat{\alpha})^{1/2}$. □

Example 3.1 *Let A be singular and B=C=I. Then the conditions of Proposition 3.2 are met with $\hat{\alpha} = -1$ and $\bar{\alpha} \neq -1$.*

Example 3.2 *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

For most values of L we get that all the DAEs (33) are index one. For sufficiently smooth f the solution will be continuous in f. However, for L equal say π we have that (33),(36) will be index two for $j = 2$ and index one for $j \neq 2$. Thus ν_M^∞ is 1 or 2 depending on L.

3.5 Comparison of Indices

We now will compute the algebraic and perturbation indices for Example 3.2. An easy calculation gives

$$R(s,z) = 4 \begin{bmatrix} \frac{4+x^2}{16s+4sz^2-z^4} & \frac{z^2}{16s+4sz^2-z^4} \\ \frac{z^2}{16s+4sz^2-z^2} & \frac{4s}{16s+4sz^2-z^4} \end{bmatrix} \tag{37}$$

Only the $R_{2,2}$ term is of interest since all the other terms are proper. We have $\nu_x = 0$, $\nu_t = 1$, and $\nu_A^{\infty} = 1$.

The modal DAE is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_j' + \begin{bmatrix} 0 & \gamma \\ \gamma & 1-\gamma \end{bmatrix} u_j = f_j \tag{38}$$

where $\gamma = \frac{j^2\pi^2}{4L^2}$. Note that $lim_{j\to\infty}\gamma = +\infty$. For a given value of L there will be at most one exceptional case, namely when $\gamma = 1$. Other than for this case, if it exists, simple algebra gives (for $\gamma \neq 1$) that (38) is the same as

$$u_{1j}'(t) - \frac{\gamma^2}{1-\gamma}u_{1j} = \frac{\gamma}{1-\gamma}f_{2j} + f_{1j} \tag{39a}$$

$$u_{2j} = \frac{1}{1-\gamma}[f_{2j} - \gamma u_{1j}] \tag{39b}$$

The key point to note from the index one system (39) is that the solutions look like bounded functions of f_j as j goes to infinity since $-\gamma^2/(1-\gamma)$ is positive for large j. Thus we have that with respect to the norms given

Proposition 3.3 *If the length L is such that $\gamma \neq 1$ for any j, then all three indices ν_M^{∞}, ν_P^{∞}, ν_A^{∞} of the PDAE in Example 3.2 are one. On the other hand if $\gamma = 1$ for some value of j, then $\nu_A^{\infty} = 1$, $\nu_M^{\infty} = \nu_P^{\infty} = 2$.*

This example illustrates an important point, namely that the algebraic and the perturbation index can differ. The next proposition gives a more general result.

Proposition 3.4 *Suppose that A is a singular square matrix. Then there is an open set of matrices B, C such that*

(i) *$sA + z^2B + C$ is regular*

(ii) *The modal DAEs are all index one for most L. In this case $\nu_M^{\infty} = 1$. If the modal DAE have negative eigenvalues, then $\nu_P^{\infty} = 1$ also.*

(iii) *There is a sequence of L, with no accumulation point, for which at least one of the modal DAEs has ν_P, ν_M greater than one.*

Proof. Without loss of generality, by performing coordinate changes we may assume that $A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Then $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$. The modal DAE will be index one precisely when $\gamma B_4 + C_4$ is invertible. Also $det(\gamma B_4 + C_4)$ is a polynomial in γ whose coefficients are continuous in the entries of B_4, C_4. The proof is completed by noting that there is an open set of B_4, C_4 such that $\gamma B_4 + C_4$ is regular and $det(\gamma B_4 + C_4) = 0$ for some $\gamma < 0$. $\square$

To help understand what is happening in this example, note that a given entry of $R(s, z)$ has the form

$$R_{i,j}(s,z) = \frac{\sum_{i=0}^{m_1} p_i(z^2)s^i}{\sum_{i=0}^{m_2} q_i(z^2)s^i} \tag{40}$$

where the p_i, q_i are polynomials. However, z^2 is really the operator $\partial^2/\partial x^2$. If α is an eigenvalue of $\partial^2/\partial x^2$ with corresponding eigenvector ϕ, then on that mode the system looks like

$$R_{i,j}(s,z)_{|z^2=\alpha} = \frac{\sum_{i=0}^{m_1} p_i(\alpha)s^i}{\sum_{i=0}^{m_2} q_i(\alpha)s^i}\phi \tag{41}$$

If $q_{m_2}(\alpha) = 0$, then it is possible for $R_{i,j}(s, \sqrt{\alpha})$ to have higher index in s then $R_{i,j}(s, z)$ does. For example, for (36) suppose that we have that -4 is an eigenvalue for $\partial^2/\partial x^2$ so that z^2 would be replaced by -4. Then we get that

$$R = 4\begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{s}{4} \end{bmatrix} \tag{42}$$

Note that (36) had t-index of 1 but (42) has t-index of 2 because of the reduction of order in terms of s of the denominator in $R_{2,2}$.

In this example we have taken the boundary conditions as part of the definition and the initial conditions as the data which may or may not be consistent. However, there are other options. For example we could view both initial conditions and the boundary conditions as data. In this case we are allowing for them to be consistent for some solutions and not consistent for other solutions.

Our next example illustrates why we had to modify the definition of the perturbation index to include derivatives of the data. This PDAE is not in the form (18).

Given a function $f(z,w)$ of several variables, let L_z be the operator of antidifferentiation with respect to z, i.e.

$$L_z(f(z,w)) = \int_0^z f(s,w)ds \tag{43}$$

Example 3.3 *Consider the PDAE*

$$\begin{aligned}
u_t + v_{xx} &= f_1(x,t) && (44a)\\
v_{tx} &= f_2(x,t) && (44b)\\
u(x,0) &= g_1(x) && (44c)\\
v(x,0) &= g_2(x) && (44d)\\
v(0,t) &= g_3(t) && (44e)\\
0 \le x \le 1, &\qquad 0 \le t \le T && (44f)
\end{aligned}$$

We treat (44c)-(44e) as data. The solution of the PDAE (44a) and (44b) is

$$\begin{aligned}
u &= L_t f_1 - t c_2''(x) - L_t L_t \frac{\partial f_2}{\partial x} + c_3(x) && (45a)\\
v &= L_t L_x f_2 + L_t c_1(t) + c_2(x) && (45b)
\end{aligned}$$

Applying the data to (187) we get that

$$\begin{aligned}
c_3(x) &= h_1(x)\\
c_2(x) &= h_2(x)\\
L_t c_1(t) &= h_3(t)
\end{aligned}$$

From the solution we thus see that u, v depend on first x-derivatives of f but second x-derivatives of the data g. In this case we would say that $\nu_P^\infty = 3$. We have not yet defined the modal index, ν_M^∞, for this system since the g_j defined earlier are not eigenfunctions of $\partial^2/\partial x^2 + \partial/\partial x$.

On the other hand, for this example

$$R(s,z) = \begin{bmatrix} \frac{1}{s} & -\frac{z}{s^2} \\ 0 & \frac{1}{sz} \end{bmatrix} \tag{47}$$

so that $\nu_P^\infty = 3$ while $\nu_A^\infty = 2$. Note that ν_A^∞ in this example captures the dependence on the first x-partial of f_2. However, it misses the dependence on the second x-partial of g_1.

This dependence on derivatives of data is a new type of behavior not present with finite dimensional DAEs.

One way in which PDAEs differ significantly from ODEs is that there is more than one direction in which system can exhibit higher index. In fact, an explicit PDE can have index greater than zero.

Example 3.4 *Consider the explicit PDE in the form of (18)*

$$\begin{aligned} v_t + u_{xx} &= f_1(x,t) && \text{(48a)} \\ u_t &= f_2(x,t) && \text{(48b)} \end{aligned}$$

or equivalently

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = 0, \quad D = 0 \tag{49}$$

We see in this example that all the modal DAEs are index zero so that $\nu_M^\infty = 0$. The solutions of (48) are

$$\begin{aligned} u &= L_t f_2 + c_1(x) && \text{(50a)} \\ v &= L_t\left(f_1 - L_t\frac{\partial^2 f_2}{\partial x^2} - c_1''(x)\right) + c_2(x) && \text{(50b)} \end{aligned}$$

Here $\nu_P^\infty = 3$ because the solution depends on second derivatives with respect to x of a t-integral of f_2 and also second derivatives of $c_1 = u(x,0)$. Thus in general, we can not just use the indices of the modal equations themselves to estimate the index of a PDAE. For this example, we have

$$R(s,z) = \begin{bmatrix} \frac{1}{s} & -\frac{z^2}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \tag{51}$$

The algebraic t-index is zero, the algebraic x-index is 3, and $\nu_A^\infty = 3$. Unlike the previous Example 3.3, where $\nu_A^\infty < \nu_P^\infty$, we have here that $\nu_M^\infty < \nu_A^\infty = \nu_P^\infty$.

As an additional illustration suppose (18) has the form

$$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}\begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + \begin{bmatrix} C_1 & C_2 \\ 0 & C_3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \end{bmatrix} \tag{52}$$

Notice that

$$B_3 v_{xx} + C_3 v = f_2(x,t) \tag{53}$$

is a DAE in the spatial derivative $\partial/\partial x$.

The next theorem gives a general statement that covers this example.

Theorem 3.1 *Consider the PDAE $Au_t + Bu_{xx} + Cu = f(x,t)$ with boundary conditions (19). If $\lambda B + C$ is a regular pencil with index ν, then $\nu_P^\infty \geq 2\nu - 1$.*

Proof. We take an $f(x)$ whose first ν derivatives also satisfy the boundary conditions and look for a steady state solution $u(x)$. But then $Bu'' + Cu = f$. Since $\lambda B + C$ is regular of index ν we know from the solution of the DAE that u depends on $2(\nu - 1)$ derivatives of f. □

In the modal DAE, a spatial PDAE gets converted to an algebraic equation.

Example 3.5 *Consider the PDAE*

$$\begin{bmatrix} N^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = f(x,t) \tag{54}$$

which has the solutions

$$u_1 = f_1 - f_3 + f_{4xx} \tag{55a}$$
$$u_2 = f_2 - f_4 - f_{1t} + f_{3t} - f_{4txx} \tag{55b}$$
$$v_1 = f_3 - f_{4xx} \tag{55c}$$
$$v_2 = f_4 \tag{55d}$$

This example has t-index 2, x-index 3, and $\nu_P^\infty = \nu_A^\infty = 4$, but $\nu_M^\infty = 2$.

The next example generalizes Example 3.4. It is not covered by Theorem 3.1.

Example 3.6 *Let N be an $r \times r$ upper triangular nilpotent Jordan block and consider the PDE*

$$u_t + N u_{xx} = f \tag{56}$$

The modal equations are all index zero so that $\nu_M^\infty = 0$. However,

$$u = \sum_{i=0}^{r-1} (-N L_t \frac{\partial^2}{\partial x^2})^i f + \sum_{i=0}^{r-1} \frac{t^i}{i!} (-N \frac{\partial^2}{\partial x^2})^i c(x) \tag{57}$$

where $c(x)$ is an arbitrary function of x. For this example, $\nu_M^\infty = 0$ but $\nu_P^\infty = \nu_A^\infty = 2(r-1)+1$.

Example 3.7 *Let N in Example 3.6 be an $r \times r$ upper triangular nilpotent Jordan block. Then $\nu_M^\infty = r$ but $\nu_P^\infty = \nu_A^\infty = 2r$.*

Another way that GDAEs differ from DAEs is that the index can depend on the particular way we present initial conditions. In particular, the consistent initial conditions can satisfy differential equations themselves. $\mathcal{D}$ can then be chosen to be different projections of the initial values.

Example 3.8 *Consider the PDAE in the form (18)*

$$u_t + v_{xx} = f_1(x,t) \tag{58a}$$

$$v_{xx} - w = f_2(x,t) \tag{58b}$$

$$w_t = f_3(x,t) \tag{58c}$$

along with the homogeneous boundary conditions and the two different initial conditions in (19)

$$v(x,0) = v_0(x) \tag{59}$$

or

$$w(x,0) = w_0(x) \tag{60}$$

The general solution of the PDAE (58) can be written

$$u = L_t(f_1 - f_2 - L_t f_3 - c_1(x)) + c_4(x) \tag{61a}$$

$$v = L_x L_x(f_2 + L_t f_3 + c_1(x)) + x c_2(t) + c_3(t) \tag{61b}$$

$$w = L_t f_3 + c_1(x) \tag{61c}$$

We assume that f also satisfies the initial condition in (19) which implies that c_1, c_2 are determined by the boundary conditions. Also

$$u(x,0) = c_4(x) \tag{62}$$

If we use the initial condition (59), (62), then $w(x,0) = c_1(x) = w_0(x)$ and $\nu_P^\infty = 1$. But if we use the initial condition (60), (62) we have that $v_0(x) = L_x L_x(f_2(x,0) + c_1(x)) + x c_2(0) + c_3(0)$. Thus $c_1(x) = v_0''(x)$ and $\nu_P^\infty = 3$. On the other hand, for this example we have that

$$R(s,z) = \begin{bmatrix} s & z^2 & 0 \\ 0 & z^2 & -1 \\ 0 & 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & -\frac{1}{s} & -\frac{1}{s^2} \\ 0 & \frac{1}{z^2} & \frac{1}{z^2 s} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \tag{63}$$

Thus $\nu_a^\infty = 1$. Also $\nu_M^\infty = 1$.

One of the things that ν_M^∞, ν_A^∞ fail to capture is the need for convergence of the approximate series. The next example illustrates this.

Example 3.9 *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{64}$$

The modal DAE is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_j' + \begin{bmatrix} 0 & \gamma \\ -\gamma & 1-\gamma \end{bmatrix} u_j = f_j \tag{65}$$

where $\gamma = \frac{j^2\pi^2}{4L^2}$. For a given value of L there will be at most one exceptional case, namely when $\gamma = 1$. Other than for this case if it exists, simple algebra gives (for $\gamma \neq 1$) that (65) is the same as

$$u_{1,j}'(t) + \frac{\gamma^2}{1-\gamma}u_{1j} = \frac{\gamma}{1-\gamma}f_{2j} + f_{1j} \tag{66a}$$

$$u_{2j} = \frac{1}{1-\gamma}[f_{2j} + \gamma u_{1j}] \tag{66b}$$

Let $f = 0$ for the moment. Then we have that

$$u_{1j} = e^{-t\gamma^2/(1-\gamma)}u_{1j}(0) \tag{67}$$

But $\gamma \to \infty$ as $j \to \infty$. Thus ν_P^∞ is not even defined unless we place strong assumptions on the convergence of the Fourier coefficients of both the initial conditions and on f.

Since the exponential also appears in the solution for $f \neq 0$,

$$\int_0^t e^{-(s-t)\gamma^2/(1-\gamma)}f(s)ds \tag{68}$$

we get that an data that may be acceptable on one time interval may not be acceptable on a longer time interval. In this case we could have a dependence of ν_P^∞ on the length of the time interval.

We now will compute the algebraic indices. An easy calculation gives

$$R(s,z) = 4\begin{bmatrix} \frac{4+z^2}{16s+4sz^2+z^4} & \frac{z^2}{16s+4sz^2+z^4} \\ \frac{-z^2}{16s+4sz^2+z^4} & \frac{4s}{16s+4sz^2+z^4} \end{bmatrix} \tag{69}$$

Only the R_{22} term is of interest since all the other terms are proper. We have $\nu_x = 0$, $\nu_t = 1$, and $\nu_A^\infty = 1$.

Summarizing the above, we have shown how to define various indices for the linear PDAE. Examples show that even for linear time invariant PDAEs all three indices can differ from each other. The perturbation index may depend on inputs, data (initial conditions) and on the domain.

3.6 Examples from Applications

We now mention some examples of PDAEs that arise in applications. More examples can be found in [14]. We do not present a detailed analysis of these examples. Rather our intention is to illustrate how PDAEs of types discussed earlier occur and what are their indices.

Example 3.10 *The incompressibe Navier-Stokes equations take the form*

$$\begin{aligned} \frac{\partial u}{\partial t} + (u.\nabla)u + \nabla p - \gamma \nabla^2 u &= 0 && (70a) \\ \nabla . u &= 0 && (70b) \end{aligned}$$

Here u is a 3-vector and p is a scalar function of 3 variables. For our purposes we shall give the 1-D form of these equations. Let $u = [u_1,\ p]$. Then we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_t - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_{xx} + \begin{bmatrix} u_1 & 1 \\ 1 & 0 \end{bmatrix} u_x = 0 \qquad (71)$$

The linearized form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_t - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_{xx} + \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix} u_x = f \qquad (72)$$

with f=0. For this system

$$R(s,z) = \begin{bmatrix} s + \alpha z - \gamma z^2 & z \\ z & 0 \end{bmatrix}^{-1} = \frac{1}{z^2} \begin{bmatrix} 0 & z \\ z & \gamma z^2 - \alpha z - s \end{bmatrix} \qquad (73)$$

so that the t-index is 2 and the x-index is 1. Letting f be nonzero and solving we get that

$$\begin{aligned} u_1 &= L_x f_2 + \phi(t) && (74a) \\ u_2 &= -L_x L_x \frac{\partial f_2}{\partial t} - \phi'(t) + \psi(t) - \alpha L_x f_2 + \gamma f_2 + f_1 && (74b) \end{aligned}$$

where $\phi(t)$, $\psi(t)$ are arbitrary functions that are determined by the boundary consitions. The presence of $\phi'(t)$ and $(f_2)_x$ shows the perturbation index is two for some boundary conditions. This reflects the well known fact that the MOL solution of the Navier-Stokes equation leads to an index two DAE [1].

Example 3.11 *Consider a tubular reactor described by the following system [26]*

$$\begin{aligned} C_t &= c_1 C_{zz} + c_2 C_z + c_3 C && (75a) \\ h_t &= c_4 T_{zz} + c_5 h_z + c_6 C && (75b) \\ 0 &= -h + c_7 T + c_8 T^2 + c_9 T^3 + c_{10} T^4 && (75c) \end{aligned}$$

Note that while it is theoretically possible to solve (75c) for T it is not practical to do so and the problem is best handled as a DAE. The constraint (75c) is nonlinear. Let $u = [C, h, T]^T$. If we linearize we get locally

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_t - \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & c_4 \\ 0 & 0 & 0 \end{bmatrix} u_{zz} - \begin{bmatrix} c_2 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_z - \begin{bmatrix} c_3 & 0 & 0 \\ c_6 & 0 & 0 \\ 0 & -1 & c_{11} \end{bmatrix} u = 0 \quad (76)$$

None of the coefficients is invertible. As noted earlier, the algebraic index will sometimes indicate parameter values resulting in higher index behavior. Let $\Delta_1 = s - c_2 z - c_3 - z^2 c_1$ and $\Delta_2 = sc_{11} - c_{11}c_5 z - c_4 z^2$. Then

$$R(s,z) = \frac{1}{\Delta_1 \Delta_2} \begin{bmatrix} \Delta_2 & 0 & 0 \\ c_6 c_{11} & c_{11}\Delta_1 & -c_4 z^2 \Delta_1 \\ c_6 & \Delta_1 & (s - c_5 z)\Delta_1 \end{bmatrix} \quad (77)$$

For most values of the c_i we get algebraic index 1 in t and x. However, if $c_{11} = 0$ but $c_4 \neq 0$, then the (3,3) entry has t-index equal 2. The perturbation index is also 2 for some boundary conditions if $c_{11} = 0$, and $c_4 \neq 0$ since we then have that

$$u_2 = f_3 \quad (78a)$$

$$u_3 = \frac{1}{c_4} L_z L_z \left(\frac{\partial f_3}{\partial t} - c_5 \frac{\partial f_3}{\partial z} - c_6 u_1 - f_2 \right) + \phi_1(t) + \phi_2(t) \quad (78b)$$

with $u_1 = C$.

Having c_{11} small but nonzero corresponds to a situation where the DAE that results from the method of lines is index 1 but the Jacobian of the constraint is ill conditioned and numerical difficulties are possible since the index 1 solution is trying to approximate the index 2 solution when $c_{11} = 0$.

4 Differential Algebraic Equations in Infinite Dimensional Linear Systems

4.1 Linear Partial Differential Algebraic Equations: Method of Lines and Modal Analysis

We shall consider here the analysis of a linear PDAE (18), (19) via the method of lines (MOL) and compare the results with the modal approach discussed in Subsection 3.5.

The MOL solutions of PDAEs will be considered here for both consistent and inconsistent data (initial and boundary conditions). We will show that for the inconsistent data, the MOL solution of a DAE can be considered as an approximate solution of the corresponding PDAEs with impulsive terms. This applies also to the boundary control problem considered in section 4.3.

There has been a substantial amount of effort expended in the numerical analysis communities in developing numerical methods for integrating DAEs. One of the earliest motivations for developing DAE software was that DAEs occur in the MOL. Even when the PDE is explicit, it usually happens that adding various constraints or boundary controls, the problem becomes implicit. Usually MOL is used without worrying about whether the DAE being solved correctly reflects the PDAE. We will show how our discussion of fully implicit PDEs has important implications for the interpretation of MOL simulation of PDAEs.

It suffices for our purpose to consider (18),(19) with $D = 0$ in (t, x) on rectangular domain. By MOL we will mean a finite dimensional approximation in the x variable, by either finite differences or finite elements, leaving a DAE in t.

4.1.1 MOL Using Differences

Let Q be the $N \times N$ tridiagonal matrix with $Q_{i,i} = a$, $Q_{i,i+1} = b$, $Q_{i+1,i} = c$. The N eigenvalues of Q are given in [32]. For the special case of $a = -2$, $b = c = 1$, the eigenvalues and eigenvectors are

$$\hat{\lambda}_s = -2 + 2cos\frac{s\pi}{N+1},\ v_s = [\sin\frac{s\pi}{N+1}, \sin\frac{2s\pi}{N+1}, \sin\frac{3s\pi}{N+1}, ..., \sin\frac{Ns\pi}{N+1}]^T;\ s = 1, ..., N. \tag{79}$$

B We also need the following basic properties of the $\otimes$ product of two matrices (the Kronecker product). For any two matrices A, B the product $A \otimes B$ replaces each entry of A by $a_{i,j}B$. Thus $(A \otimes B)(C \otimes D) = AC \otimes BD$, and $(A \otimes B)^T = A^T \otimes B^T$ if the matrices are of correct sizes. Let $\bar{D} = I \otimes D$ be a block diagonal matrix with D on the diagonal so that $\bar{D}(A \otimes B) = A \otimes DB$. If α is a scalar, then $\alpha(A \otimes B) = (\alpha A \otimes B) = (A \otimes \alpha B)$.

Suppose that we are doing MOL with $N + 1$ evenly spaced grid points $x_s = sL/(N + 1)$

using centered differences. Let $U_i(t) = u(x_i, t)$. The MOL DAE is then $AU_i' + \frac{1}{h^2}B(U_{i+1} - 2U_i + U_{i+1}) + CU_i = \hat{f}_i$, $1 \le i \le N$, where $\hat{f}_i = f(x_i, t)$ and U_0, U_N are zero if $u(0,t) = u(L,t) = 0$. Letting U be the vector of the U_i for $i = 1, ..., N$, gives

$$\bar{A}U' + \frac{1}{h^2}\bar{B}MU + \bar{C}U = \hat{f} \tag{80}$$

where $M = P \otimes I_n$, I_n is an $n \times n$ identity matrix, the same size as A, B, C and P is tridiagonal with $\alpha = -1$ and $b = c = 1$.

Let ε be the evaluation map at the N interior grid points defined by $\varepsilon(p(x)) = [p(x_1)^T, ..., p(x_N)^T]^T$. Note that ε depends on the (fixed) value of N and $\varepsilon(f) = \hat{f}$. For any positive integer n there is a positive integer j and integer β with $-N - 1 \le \beta < N + 1$ such that $n = 2j(N+1) + \beta$. A straightforward calculation show that

$$\varepsilon(\phi_n(x)) = \begin{cases} v_\beta & \text{if } N+1 > \beta > 0 \\ -v_{|\beta|} & \text{if } -(N+1) < \beta < 0 \\ 0 & \text{if } \beta = 0, -N-1 \end{cases} \tag{81}$$

Thus $\varepsilon(f) = \sum_{s=1}^N v_s \otimes g_s$, $g_s(t) = \sum_{q=1}^\infty (-1)^{q-1} f_{s+q(N+1)}(t)$ and $\varepsilon(\sum_{s=1}^N u_s(t)\phi_n(x)) = \sum_{s=1}^N v_s \otimes u_s(t)$. The solution U of (80) can be written $U = \sum_{s=1}^N v_s \otimes w_s(t)$ where w_s is an unknown n-vector function. Substituting U into (80) and using the properties of the Kronecker product gives

$$\sum_{s=1}^N v_s \otimes Aw_s'(t) + \frac{1}{h^2}\sum_{s=1}^N Pv_s \otimes Bw_s(t) + \sum_{s=1}^N v_s \otimes Cw_s(t) = \hat{f}. \tag{82}$$

But $Pv_s v_s = \hat{\lambda}_s$. Multiplying by the matrices $v_i^T \otimes I$ and using the orthogonality of the v_s, which holds since they come from a symmetric matrix, we get finally the DAEs

$$Aw_s' + (\frac{1}{h^2}\hat{\lambda}_s B + C)w_s = g_s, \qquad s = 1, ..., N \tag{83}$$

Notice that the approximation to the true solution using the first N modes under ε becomes $\varepsilon(\sum_{n=1}^N \phi_n u_i(t)) = \sum_{n=1}^N v_n \otimes u_i(t)$ so that modal expansions of the solutions of the PDAE are mapped to eigenfunction expansions of the MOL DAE. A calculation gives

$$\frac{1}{h^2}\hat{\lambda}_s = \frac{-2 + 2\cos\frac{hs\pi}{L}}{h^2} = -\frac{s^2\pi^2}{L^2} + O(h^2)R(\frac{s\pi}{L})^4 = -\lambda_s^2 + O(h^2)R(\frac{s\pi}{L})^4, \tag{84}$$

and R is independent of N. Thus for a fixed s, the term $h^{-2}\hat{\lambda}_s$ converges as $h \to 0$ to the "eigenvalue" $-\lambda_s^2$ of the continuous case. However, the convergence is not uniform.

Several observations can be drawn from the above. First, suppose that PDAE has index ν_0 (based on a modal expansion). If the domain is close to a domain which would make the PDAE have index greater than ν_0, then the MOL DAE might have index greater than ν_0 for some Δx. Second, if the PDAE is index one, then the MOL DAE will also be index one for large enough N. Third, if the pencil $\lambda A + (\gamma B + C)$ has the same index for all γ, then the PDAE and the MOL DAE will have the same index. Fourth, suppose that the PDAE is index one for some L but that we have an L where it is index two. Then the MOL DAE will always be index one for sufficiently large N. The reason is that in the estimate (84) we actually have that $\lambda_s - \hat{\lambda}_s \neq 0$. An analysis of the modal DAE will not correctly reflect the index of the PDAE. Fifth, suppose that N is fixed. Suppose also that the DAEs $Au_n' + (-\lambda_n^2 B + C)u_n = f_n$, $n \geq 1$, is index two for $n = 2$ but the other equations are index one. Suppose that $f(x,t) = r(t)\phi_{N+3}(x)$. Then in the modal problem we have all the $f_i = 0$, and the solution is zero if $g = 0$. However, $\varepsilon(r(t)\phi_{N+3}(x)) = v_2 \otimes r(t)$ which is exactly the same as if we forced the problem with $r(t)\phi_2(t)$. That is, the high frequency forcing of problems with higher index low frequency modes could experience numerical difficulty with MOL as if that low frequency mode has been forced.

4.1.2 MOL with Elements

Suppose now that our approximation is $u(x,t) \approx \sum_{i=0}^{N+1} \psi_i(x) U_i(t)$ where the $\psi_i(x)$ are the scalar basis elements and the U_i are vectors. Assume that the basis elements satisfy $\psi_0(0) = 1$, $\psi_i(0) = 0$ for $i > 0$, and $\psi_{N+1}(L) = 1$, $\psi_i(L) = 0$ for $i < N+1$. The boundary conditions are incorporated by taking $U_0 = U_{N+1} = 0$. Let $< \beta, \gamma > = \int_0^L \beta(x)\gamma(x)dx$. The element DAE is calculated from $< Au_t + Bu_{xx} + Cu - f, \psi_i > = 0$, $1 \leq i \leq N$. Letting U be the vector of the U_i we get the DAE

$$(M \otimes A)U' + [-(S \otimes B) + (M \otimes C)]U = \hat{f}, \tag{85}$$

where $\hat{f}$ is the vector of $f_i = < f, \psi_i >$ and $M_{ij} = < \psi_j, \psi_i >$, $S_{ij} = < \psi_j', \psi_i' >$, $1 \leq i,j \leq N$. The index of the DAE (85) is determined by the index of the pencil

$$\{(M \otimes A), -(S \otimes B) + (M \otimes C)\}. \tag{86}$$

Proposition 4.1 *Suppose that $M^{-1}S$ is diagonalizable. Then the index of the pencil (86) is the maximum of the indices of the pencils $\{A, \gamma B + C\}$ where γ is an eigenvalue of $-M^{-1}S$.*

Proof. Suppose that $-M^{-1}S = Q\Gamma Q^{-1}$ where Γ is diagonal. Multiply the pencil on the left by $(QM^{-1}) \otimes I$ and on the right by $Q^{-1} \otimes I$ to get the pencil $\{I \otimes A, (\Gamma \otimes B) + (I \otimes C)\}$. This is a direct sum of the pencils $\{A, \gamma_i B + C\}$ where γ_i is an eigenvalue of $-M^{-1}S$. □

If linear elements are used, M and S commute so that $M^{-1}S$ is diagonalizable. It is possible that there exists other choices of elements for which $M^{-1}S$ is not diagonalizable. If this happens then there could exist PDAEs for which the MOL DAE given by elements is higher index than the original PDAE was.

Let $\overline{\nu}$ ($\underline{\nu}$) be the maximum (minimum) of the index of $\{A, \alpha B + C\}$ for $\alpha < 0$. If $\underline{\nu} < \overline{\nu}$ it is possible that the MOL DAEs using elements and differences can have different index on the same PDAE. The MOL DAEs using elements can have different index for different choices of basis elements.

4.2 Computational Examples

In this section we examine what happens computationally for some of the examples mentioned in the previous section. All computations here deal with the finite differences and the MOL. The DAEs that arise from the MOL are integrated by a first order BDF (backward Euler).

Example 4.1 *Consider the PDAE with coefficient matrices*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \tag{87}$$

with $L = \pi$. *The MOL DAE is index one for all* $\Delta x \neq \sqrt{2}$. *The modal (Fourier) DAEs are index one if* $j \neq 1$ *and index two if* $j = 1$.

(a) Let $f(x,t) = [t,t]^T \sin 2x$. This solution should not excite any impulsive modes and the computation gave a reasonable solution.

(b) Consider $f(x,t) = [e^{-t}, e^{-t}]^T \sin 2x$. This problem has inconsistent initial conditions. For the MOL DAE which is index one we theoretically get a jump condition. But the PDAE actually has an impulsive solution in the u_2 variable. Figs.1 and 2 show the solutions. Note that the index one MOL DAE is actually providing an approximation for the PDAE which is really index two.

Example 4.2 *In this series of computations we consider the PDAE (87). This PDAE is of special interest because the scalar index is higher than either the spatial index. The MOL DAE is index two while the PDAE is index four. The difference is due to spatial derivatives.*

(a) Let $f_i(x,t) = t \sin x$ for $i = 1,2,3,4$. Then $u_1 = -t \sin x$, $u_2 = \sin x$, $v_1 = 2t \sin x$, $v_2 = t \sin x$. $u_2 = 0$ is inconsistent so there is a jump at $t = 0^+$. The solutions are shown in Figs.3-6.

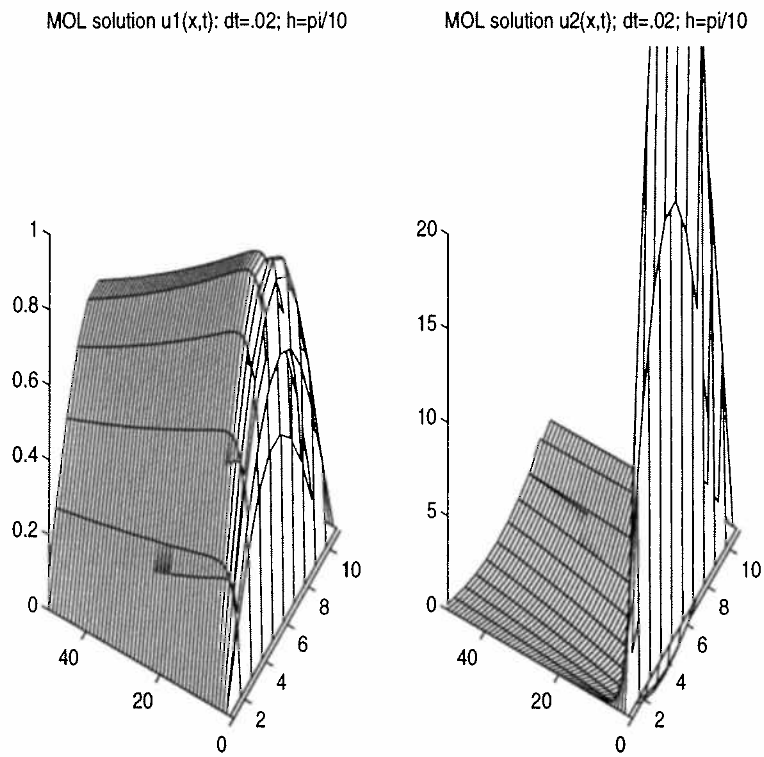

Fig.1. 3D soultions of Example 4.1, part (b).

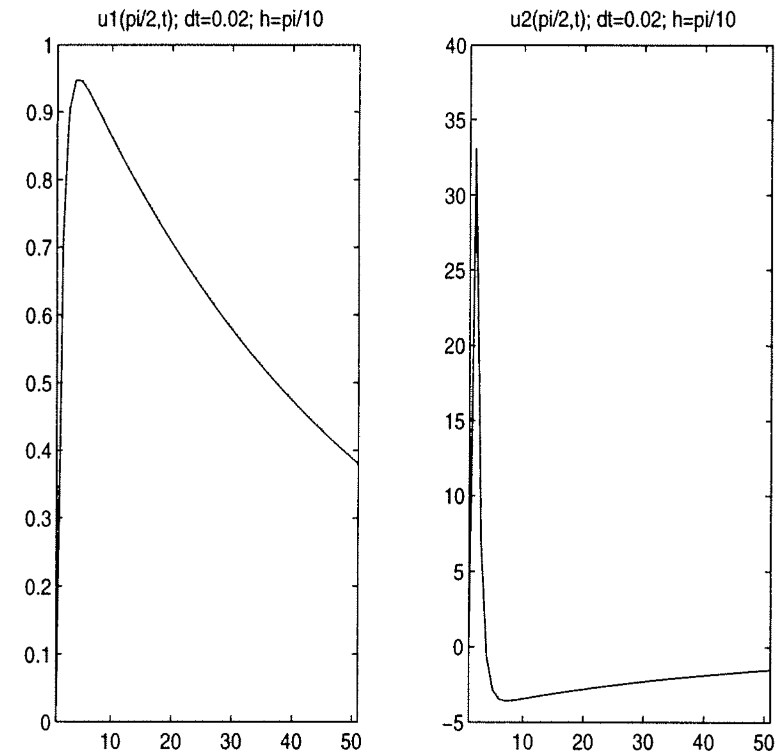

Fig.2. u_1 and u_2 for $x = \pi/2, 0 \leq t \leq 1$.

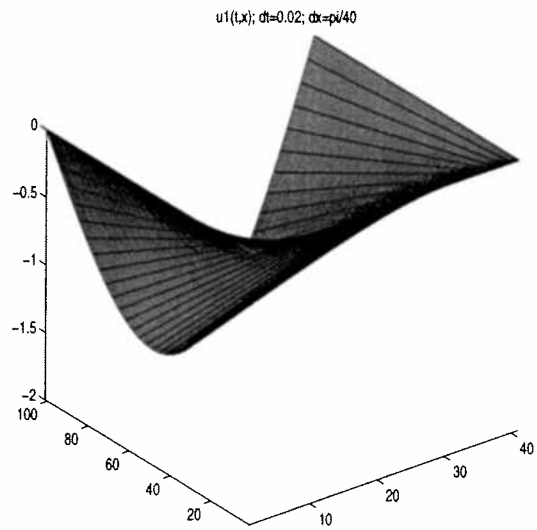

Fig.3. 3D solution for u_1 in Example 4.2, part (a)

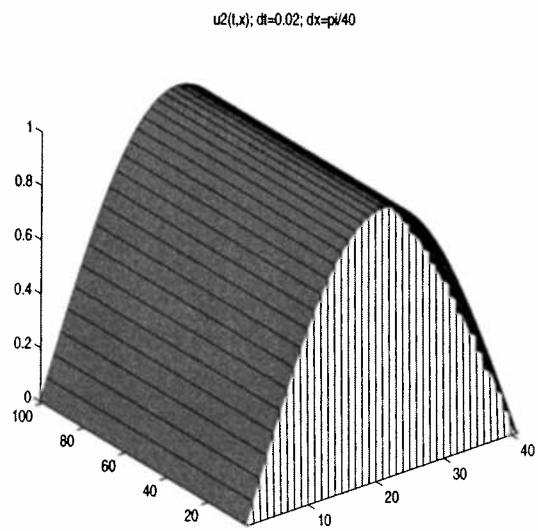

Fig.4. 3D solution for u_2 in Example 4.2, part (a)

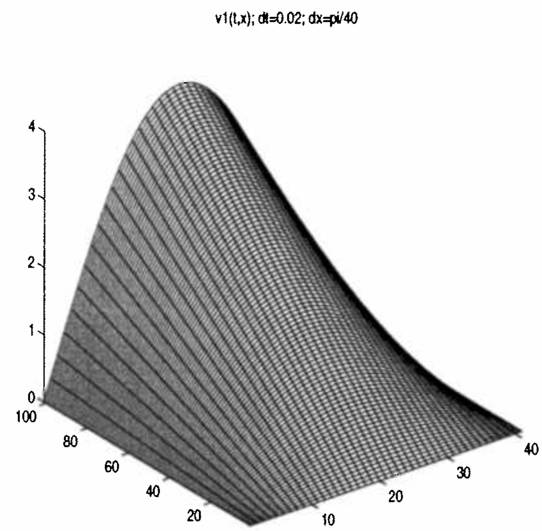

Fig.5. 3D solution for v_1 in Example 4.2, part (a)

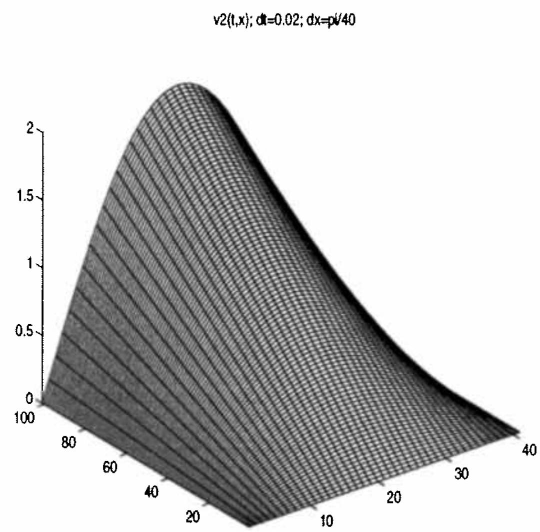

Fig.6. 3D solution for v_2 in Example 4.2, part (a)

(b) Let $f_4 = tx(x - \pi)$. Then $u_1 = 2t$, $u_2 = t\sin x - tx(x - \pi) - 2$, $v_1 = t\sin x - 2t$, $v_2 = tx(x - \pi)$. There are jumps at $x = 0^+$, $x = \pi^-$, and $t = 0^+$ in different variables. The 3-d solutions are shown in Figs.7-10, and the 2-d solutions for several time instants and $x \in [0, \pi]$ are shown in Figs.11-14. The inconsistency in the boundary conditions is due to the x derivatives.

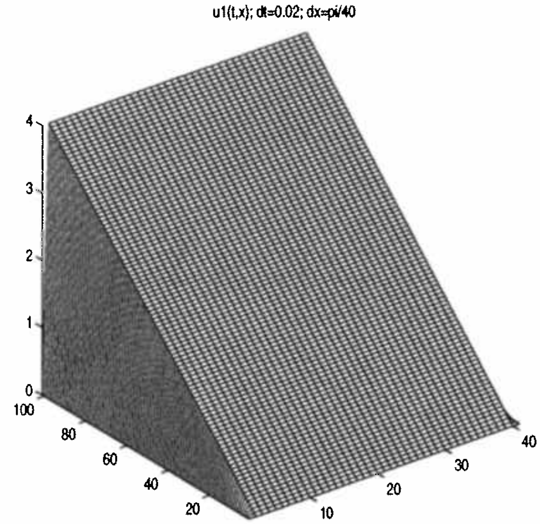

Fig.7. 3D solution for u_1 in Example 4.2, part (b)

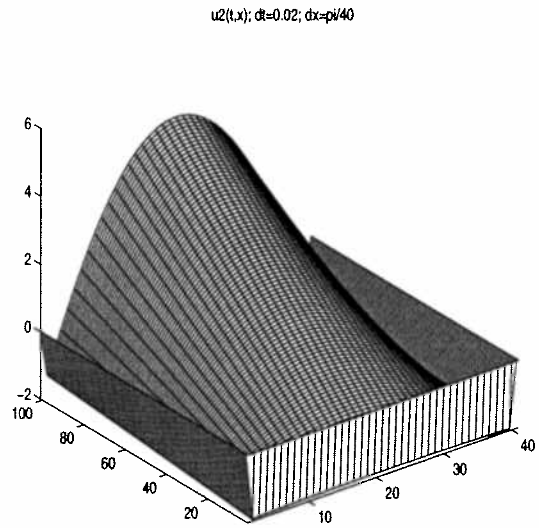

Fig.8. 3D solution for u_2 in Example 4.2, part (b)

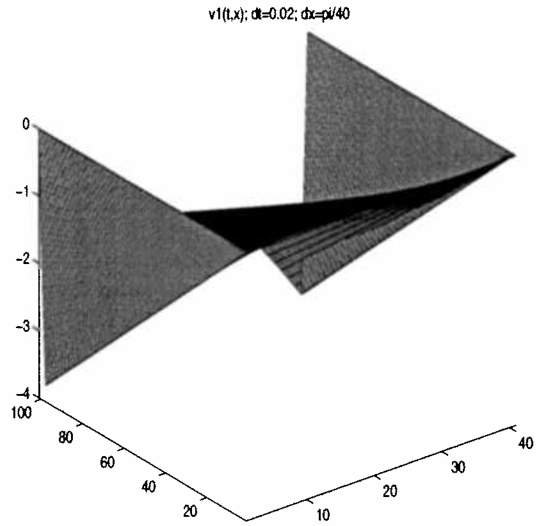

Fig.9. 3D solution for v_1 in Example 4.2, part (b)

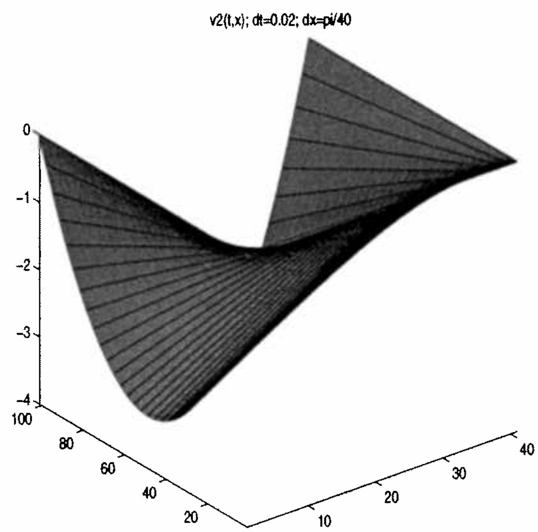

Fig.10. 3D solution for v_2 in Example 4.2, part (b)

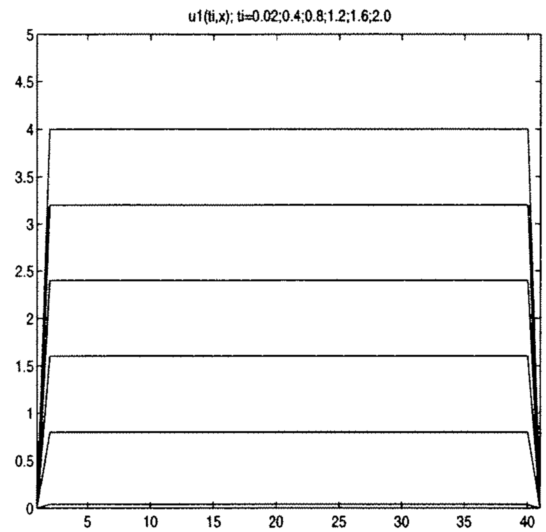

Fig.11. u_1 at different times

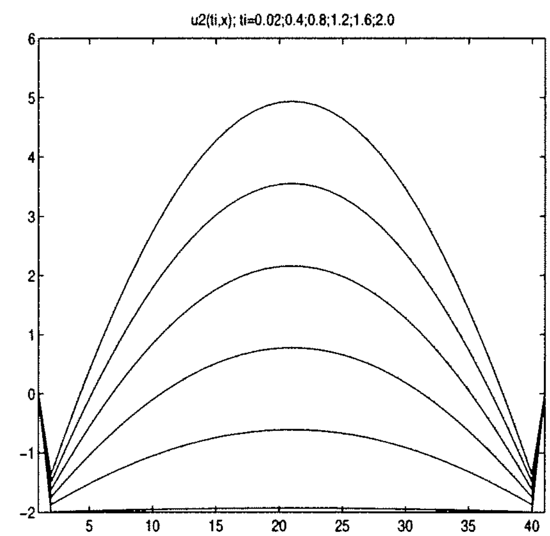

Fig.12. u_2 at different times

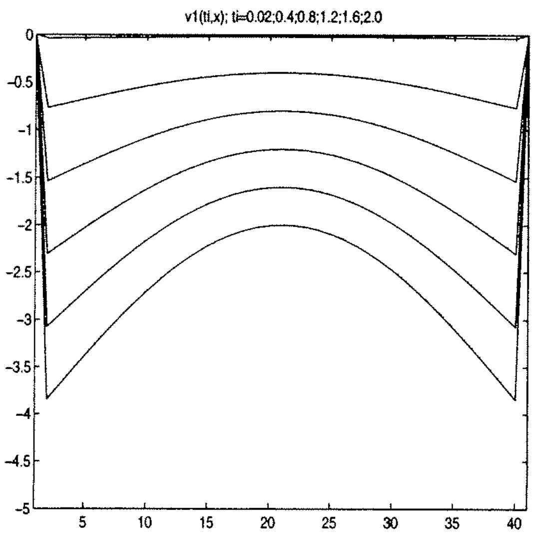

Fig.13. v_1 at different times

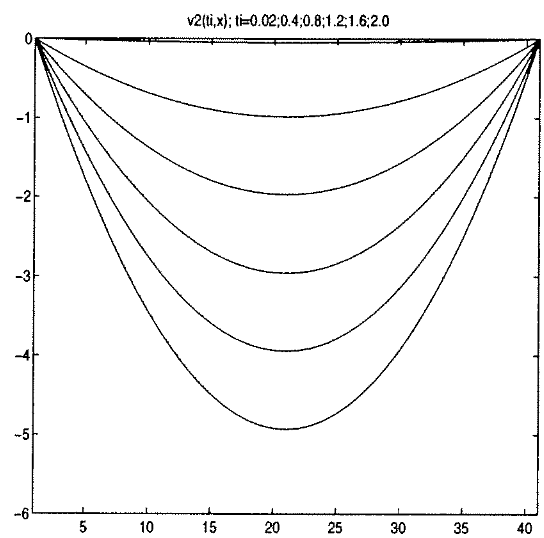

Fig.14. v_2 at different times

(c) We take $f_1 = f_2 = f_3 = (t-1)\sin x$, $f_4 = (t-1)x(x-\pi)$. Then $u_1 = 2(t-1)$, $u_2 = (t-1)\sin x - (t-1)x(x-\pi) - 2 - \delta(t)$, $v_1 = (t-1)\sin x - 2(t-1)$, $v_2 = (t-1)x(x-\pi)$. Boundary conditions are inconsistent. There are jumps at $x = 0^+$, $x = \pi^-$, $t = 0^+$ in different variables. Impulse in u_2 is due to f_{4txx}. The solutions are shown in Figs.15-22.

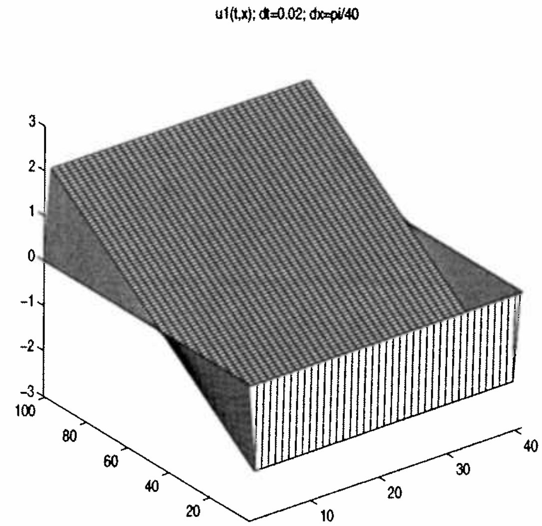

Fig.15. 3D solution for u_1 in Example 4.2, part (c)

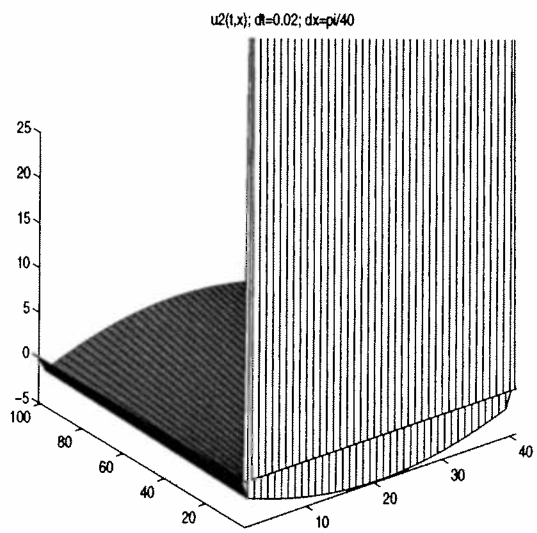

Fig.16. 3D solution for u_2 in Example 4.2, part (c)

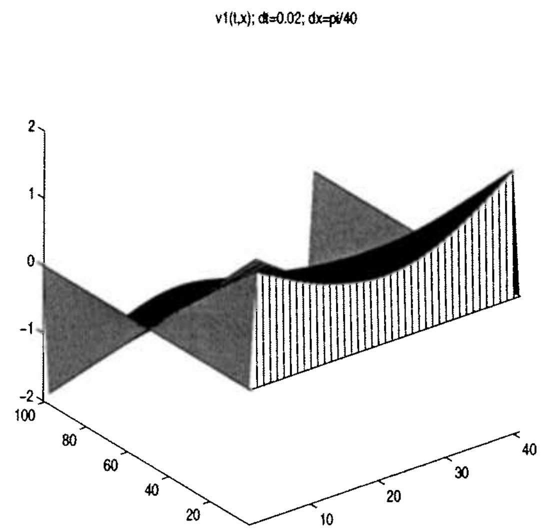

Fig.17. 3D solution for v_1 in Example 4.2, part (c)

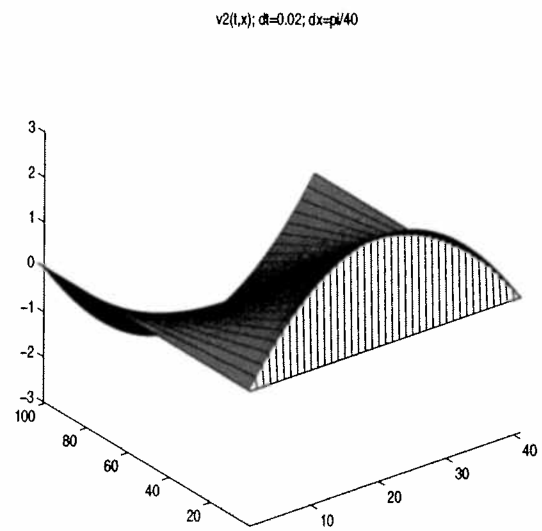

Fig.18. 3D solution for v_2 in Example 4.2, part (c)

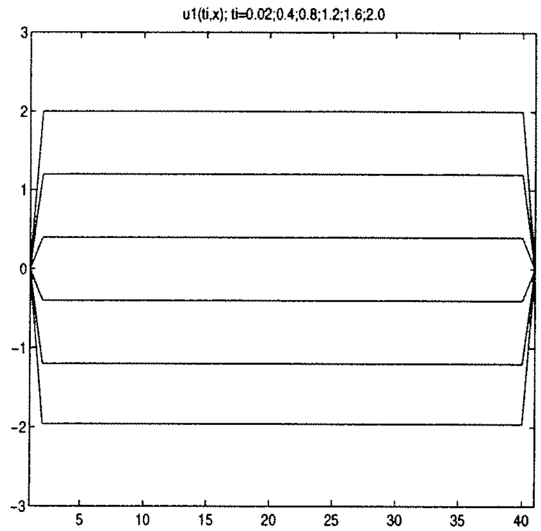

Fig.19. u_1 at different times

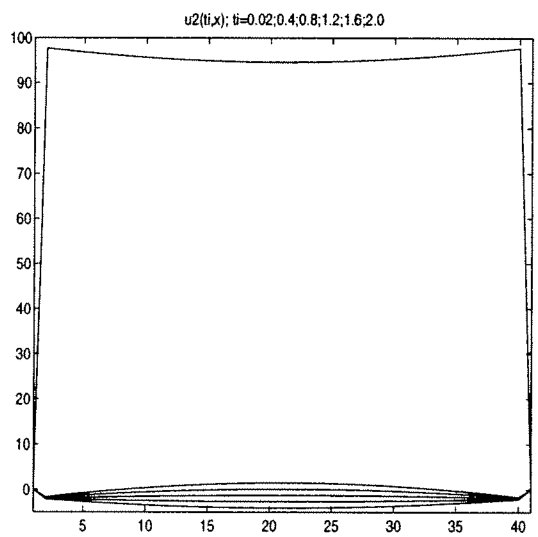

Fig.20. u_2 at different times

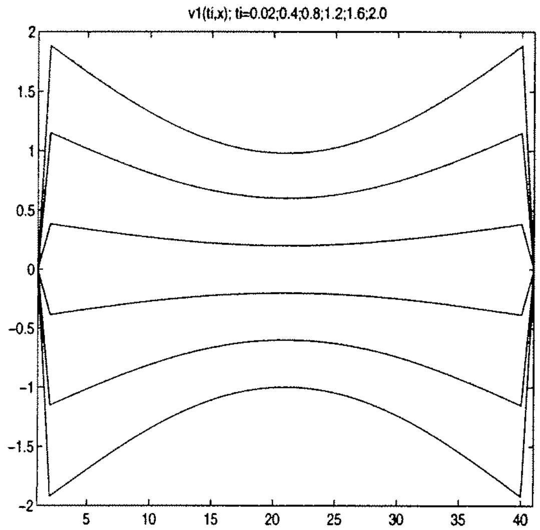

Fig.21. v_1 at different times

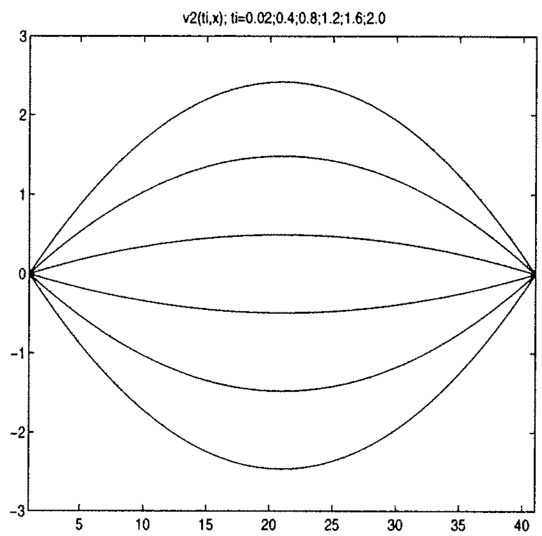

Fig.22. v_2 at different times

(d) Let $f_4 = (t-1)(x-\pi)(x+\pi)$. This produces an impulse in x. The solutions are shown in Figs.23-30.

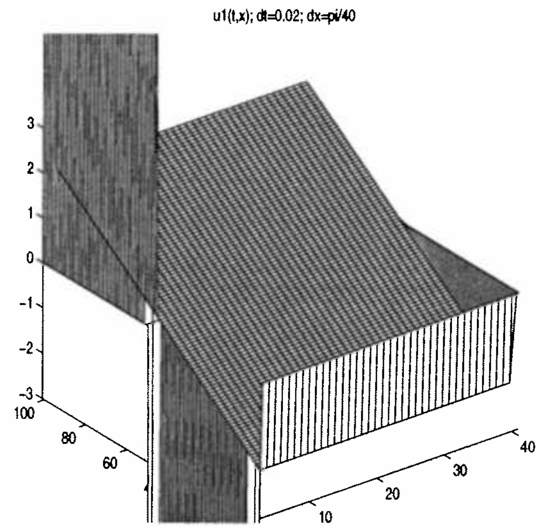

Fig.23. 3D solutions for u_1 in Example 4.2, part (d)

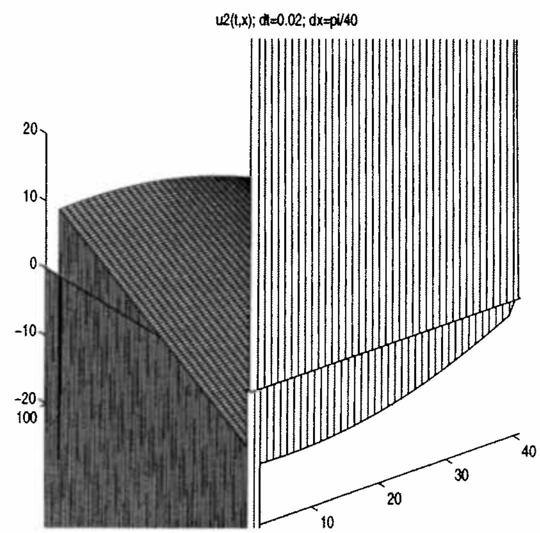

Fig.24. 3D solutions for u_2 in Example 4.2, part (d)

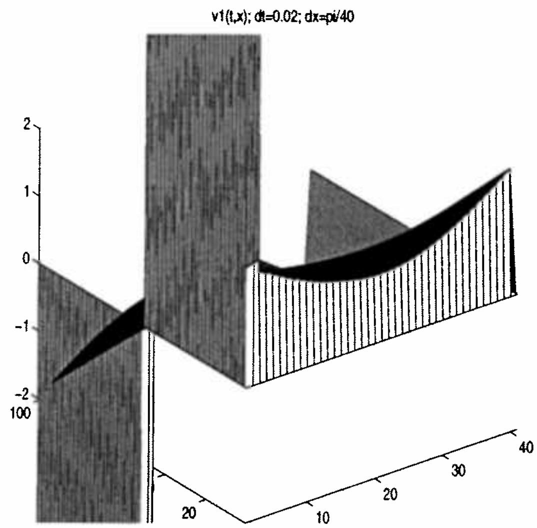

Fig.25. 3D solutions for v_1 in Example 4.2, part (d)

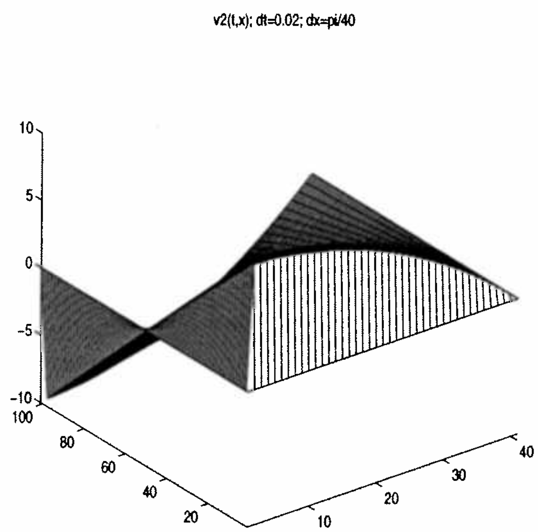

Fig.26. 3D solutions for v_2 in Example 4.2, part (d)

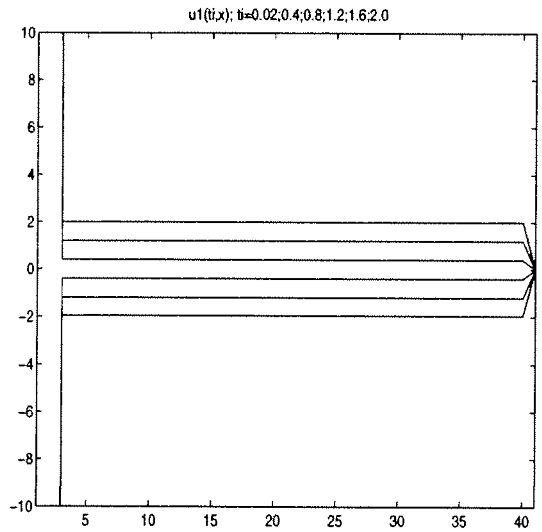

Fig.27. u_1 at different times

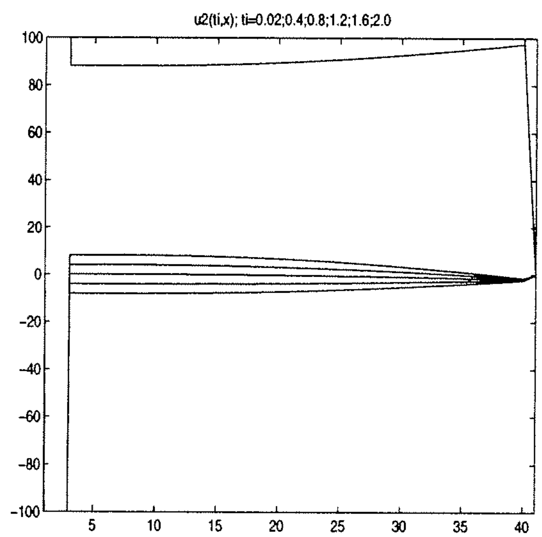

Fig.28. u_2 at different times

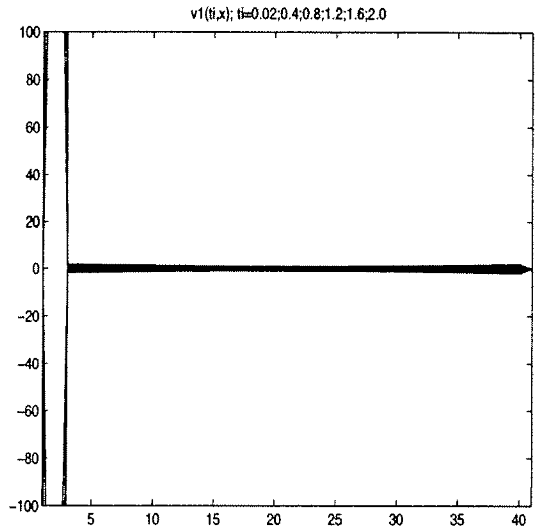

Fig.29. v_1 at different times

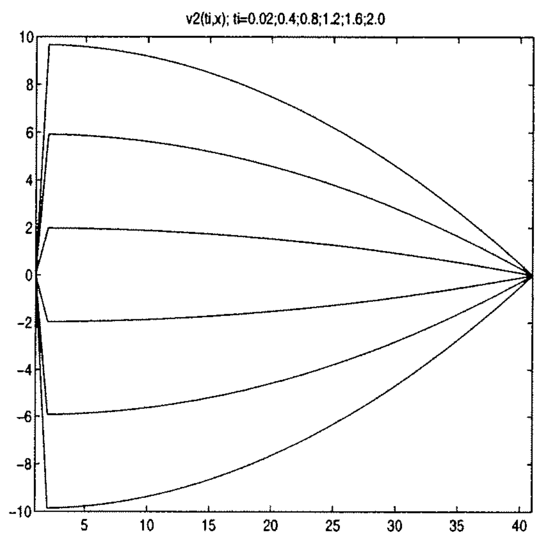

Fig.30. v_2 at different times

4.3 Boundary Control Problem

The initial formulation of this boundary control problem is due to Orlov [25] . This infinite dimensional problem has a well defined index. The whole problem is a mixture of differential, algebraic and integral equations.

The problem is

$$u_t = u_{xx} \quad t \geq 0, 0 \leq x \leq L \tag{88a}$$

$$u(0,t) = v(t) \tag{88b}$$

$$u(L,t) = 0 \tag{88c}$$

$$y(t) = \int_0^L u(x,t)p(x)dx \tag{88d}$$

We wish to consider y as a given or measured quantity and we are interested in the v, u that produce this y. In this setting (88) is a type of PDAE in the unknown $(u(x,t), v(t))$. Suppose we are interested in finding the index of (88). Note that if $p(x) = \delta(x - \frac{L}{2})$, then (88) is the problem examined in [11].

We assume that $p(x) > 0, 0 < x < L$ and if we differentiate (88d) once with respect to t, use the fact that u satisfies (88a), and then use integration by parts twice, then we obtain

$$y'(t) = \int_0^L u_t(x,t)p(x)dx \tag{89a}$$

$$= \int_0^L u_{xx}(x,t)p(x)dx \tag{89b}$$

$$= p(x)u_x(x,t)|_{x=0}^L - p'(x)u(x,t)|_{x=0}^L + \int_0^L u(x,t)p''(x)dx \tag{89c}$$

Proposition 4.2 *Suppose that $k \geq 1$ is an integer. Suppose that p is $2k$ times continuously differentiable and y is k times continuously differentiable. Suppose also that*

$$p^{(i)}(0) = 0, \quad i = 0, ..., 2k-2 \tag{90a}$$

$$p^{(i)}(L) = 0, \quad i = 0, ..., 2k-1 \tag{90b}$$

$$p^{(2k-1)}(0) = \alpha \neq 0 \tag{90c}$$

Then

$$y^{(k)}(t) = -p^{(2k-1)}(0)u(0,t) + \int_0^L u(x,t)p^{(2k)}(x)dx \tag{91a}$$

$$= \alpha v(t) + \int_0^L u(x,t)p^{(2k)}(x)dx \tag{91b}$$

$$\tag{91c}$$

and the system (88) has index $k+1$.

Example 4.3 *Consider the MOL approximation of the above boundary control problem. The PDAE itself has index three but the MOL DAE is only index two. With $y = e^t$ and $p(x) = x^3(L-x)^4$ we have that the PDAE has index three but the MOL DAE has index two. The result is an impulse in x. See Figs.31 and 32.*

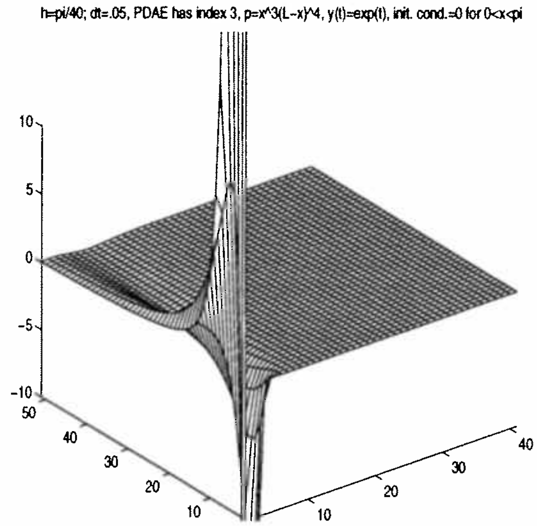

Fig.31. Solution $u(x,t)$ in Example 4.3.

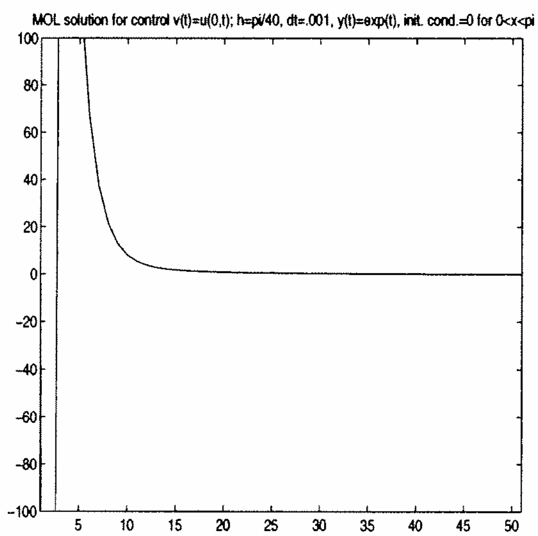

Fig.32. Boundary control $v(t)$ in Example 4.3.

4.4 Travelling and Plane Wave Differential Algebraic Equations

In this section we shall consider the PDAE of the form

$$Au_t + Bu_{xx} + Cu = 0 \tag{92a}$$

$$-\infty < x < +\infty, \qquad 0 \le t < +\infty \tag{92b}$$

$$u(x,0) = u_0(x) \tag{92c}$$

where A is singular and u_0 is twice differentiable.

We define two different wave solutions for (92). We call them *travelling wave* and *plane wave* solutions, respectively.

(a) Traveling waves

By the *travelling wave* solution of (92) we simply mean a profile $u(x-st)$ translating with the *speed* s in the (x,t) domain. Therefore, we assume that the solution of (92a), (92b) with the *initial condition* (92c) has the form $u(x,t) = u_0(x-st)$ for some $s \in \mathbf{R}$ and all $x \in (-\infty, +\infty)$ and $t \geq 0$.

Next, we introduce the following definition of what we mean by the admissibility of the travelling wave solution by system (92).

Definition 4.1 *We say that system (92) admits a family of travelling wave solutions if there exist $s_i \in \mathbf{R}$ (not necessarily distinct), scalar, twice differentiable arbitrary functions $\phi_i(z)$, and constant vectors $u_i \in \mathbf{R}^n$, $i = 1, 2, ..., m$, such that $u(x,t) = \sum_{i=1}^{m} \phi_i(x - s_i t) u_i$ is a solution of (92a) for all $x \in (-\infty, +\infty)$, $t \geq 0$.*

The above definition says that (92) admits a travelling wave solution family $u(x,t)$ if each component of the vector $u(x,t)$ is either zero or a linear combination of functions $\phi_i(x - s_i t)$, $i = 1, 2, ..., m$.

Several consequences follow from the above definition. First, we do not restrict ourselves just to one value of $s \in \mathbf{R}$ (wave speed), but allow for multiple wave speeds. Second, if the solution of (92) has the form given in Definition 4.1 for particular choices of ϕ_i and not for all twice differentiable functions, then we do not consider that (92) admits travelling wave solutions. For example, if (92) is scalar with $A = 0$, $B = 1$, $C = 1$ and $\phi(x) = sin(x)$, then the solution of (92) is $u(x,t) = sin(x - st)$ for any $s \in \mathbf{R}$. This solution is in the form $u_0(x - st)$ only because a special initial function, namely $u_0(x) = sin(x)$ was chosen. Any other initial function gives nonexistence of the solution (excluding the trivial solution). Therefore, we do not say that system $u_{xx} + u = 0$ admits a family of travelling wave solutions. On the other hand, $f(x - st)$ is a solution of $u_{xx} - s^2 u_{tt} = 0$ for any f, so this equation does admit traveling wave solutions for any s.

Proposition 4.3 *System (92) admits a family of travelling wave solutions if there exists $s \in \mathbf{R}$ such that $det(-s\lambda A + \lambda^2 B + C) = 0$ for all $\lambda \in \mathbf{C}$.*

Proof. For $u(x,t) = u_0(x - st)$ we obtain from (92a) a DAE: $-sAu_0' + Bu_0'' + Cu_0 = 0$. We can obtain all initial functions u_0, i.e. an infinite dimensional family of solutions, if the DAE has nonunique solutions. That is, the DAE is not solvable. This is equivalent to saying that $d(s,\lambda) \equiv det(-s\lambda A + \lambda^2 B + C) = 0$. $\square$

Example 4.4 *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \tag{93}$$

Then for all $\lambda \in \mathbf{C}$ we have that $det(-s\lambda A + \lambda^2 B + C) = 0$ for $s = \pm 1$ and the system admits a family of travelling wave solutions with the wave speeds ± 1.

The following observations follow immediately.

Corollary 4.1 *If $detC \neq 0$, then system (92) does not admit a family of travelling wave solutions.*

Proof. For $detC \neq 0$ we have that $det(s,\lambda) \neq 0$ for $\lambda = 0$. $\square$

Corollary 4.2 *If $C \equiv 0$ and (A, B) is a regular pencil, then (92) does not admit travelling wave solutions.*

Proof Without loss of generality we can assume that matrices A and B are in the form

$$A = \begin{bmatrix} I_{d_1} & & \\ & I_{d_2} & \\ & & N_{d_3} \end{bmatrix}, \quad B = \begin{bmatrix} N_{d_1} & & \\ & K_{d_2} & \\ & & I_{d_3} \end{bmatrix} \tag{94}$$

where I and N are the identity and nilpotent matrices, respectively, K is in Jordan form with non-zero eigenvalues at the main diagonal, and the subscripts d_i, $i = 1,2,3$ denote dimensions of matrices. Then $det(-s\lambda A + \lambda^2 B) = (-s\lambda)^{d_1}\lambda^{2d_2}\Pi_{i=1}^{q}(-s\lambda + \sigma_i\lambda^2)^{r_i}$, where q is the number of different non-zero eigenvalues $\{\sigma_1, ..., \sigma_q\}$ of K_{d_2} and r_i is the algebraic multiplicity of σ_i, $i = 1, 2, ..., q$. Therefore, $d(s,\lambda)$ can not be zero for $0 \neq s \in \mathbf{R}$ for all $\lambda \in \mathbf{C}$. $\square$

Note : If we change equation (92a) to $Au_{tt} + Bu_{xx} + C = 0$, then one can define travelling waves and their admissibility in a siimilar manner to that for (92a). Properties of the travelling wave solutions change for the new system, but generally one has to deal with similar computational aspects. Let us mention the following two properties of the later system.

Corollary 4.3 *If there exist $s_i \in \mathbf{R}$, $i = 1, 2, ..., k$, such that $d(s_i, \lambda) \equiv det(s_i^2\lambda^2 A + \lambda^2 B + C) = 0$ for all $\lambda \in \mathbf{C}$, then the system $Au_{tt} + Bu_{xx} + Cu = 0$ admits a family of travelling wave solutions with an even number of wave speeds $\pm s_i$, $i = 1, 2, ..., k$.*

Proof. Since $d(s, \lambda)$ is a polynomial with even powers of s, therefore, if $+s_i$ satisfies $d(s_i, \lambda) = 0$ for all $\lambda \in \mathbf{C}$, so does $-s_i$. $\square$

Corollary 4.4 *Let $Au_{tt} + Bu_{xx} + Cu = 0$ have regular pencil (A, B) and $C = 0$. Let $\{\bar{\sigma}_1, \bar{\sigma}_2, ..., \bar{\sigma}_{\bar{q}}\}$ be real negative (finite) roots of $det(\sigma A + B) = 0$. Then the system admits a family of travelling wave solutions with $2\bar{q}$ speeds $\pm\sqrt{-\bar{\sigma}_i}$, $i = 1, 2, ..., \bar{q}$.*

Proof. If we use canonical form (94) for $Au_{tt} + Bu_{xx} + Cu = 0$, then $d(s, \lambda) = (s\lambda)^{2d_1}\lambda^{2d_2}$ $\Pi_{i=1}^{q}(s^2 + \sigma_i)^{r_i}\lambda^{2r_i}$. Therefore, $d(s, \lambda) = 0$ for all $\lambda \in \mathbf{C}$ if there exists an $i \in \{1, 2, ..., q\}$ such that $s^2 + \sigma_i = 0$. Suppose $i = \bar{q} \le q$. Denote by $\{\bar{\sigma}_1, ..., \bar{\sigma}_{\bar{q}}\}$ all those $\sigma's$ with negative sign. The wave speeds are $s_i = \pm\sqrt{-\bar{\sigma}_i}$. $\square$

(b) Planar waves

In this section we formulate a different type of wave solution of (92a), the plane wave solution. We assume that the initial condition $u(x, 0)$ is given in the following form

$$u(x, 0) = \eta e^{i\mu x} \tag{95}$$

where $x \in (-\infty, +\infty)$.

Definition 4.2 *We say that system (92) admits a plane wave solution, if there is a solution*

$$u(x, t) = \eta e^{i(\lambda t + \mu x)} \tag{96}$$

for $x \in (-\infty, +\infty)$, $t > 0$, and the solution is bounded for all (x, t) in the domain of interest.

We are interested in finding conditions under which (96) is a bounded solution of (92a). Note that in order to get bounded $u(x, t)$ we need to restrict μ to be a real number. The λ can in general be a complex number with $Im(\lambda) > 0$.

Proposition 4.4 *If (92a) admits a plane wave solution (96) that depends continuously on the given initial condition (95), then $\frac{i\lambda}{\mu^2}$ and η are the non-zero eigenvalue and its corresponding eigenvector associated with the pencil*

$$\left\{A, -B + \frac{C}{\mu^2}\right\} \tag{97}$$

such that $Re(\frac{i\lambda}{\mu^2}) \leq 0$. *Moreover, if* $A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ *with* $det B_4 \neq 0$, *then the problem has a solution provided*

$$\frac{i\lambda}{\mu^2} = p + iq + r(\mu) + iu(\mu) \tag{98}$$

where $p + iq$ *is an eigenvalue of* $B_1 - B_2 B_4^{-1} B_3$, $p \leq 0$, *and* $r^2(\mu) + u^2(\mu) \to 0$ *as* $|\mu| \to \infty$.

Proof. Substituting (96) into (92a) gives $(Ai\lambda - B\mu^2 + C)\eta = 0$, so $s = i\lambda/\mu^2$ is a root of $det(As - B + C/\mu^2) = 0$ and η is the corresponding eigenvector. Condition $Re(i\lambda/\mu^2) \leq 0$ follows from (95) and the fact that we are looking for a bounded solution of (92a). If $|\mu| \to \infty$ and A, B have the forms given in the proposition, then $det(As - B + C/\mu^2)$ has the same degree as $det[sI - (B_1 - B_2 B_4^{-1} B_3)]$ and with the assumption that for large $|\mu|$ coefficients are continuous in $1/\mu^2$ (i.e. $lim_{|\mu|\to\infty}(i\lambda/\mu^2)$ exists) we have

$$lim_{|\mu|\to\infty} det(As - B + C/\mu^2) = det[Is - (B_1 - B_2 B_4^{-1} B_3)] det B_4 \tag{99}$$

which yields $i\lambda/\mu^2 = p + iq$, where $p + iq$ is an eigenvalue of $B_1 - B_2 B_4^{-1} B_3$. The eigenvalue of a matrix depends continuously upon the coefficients of the matrix. Therefore for any $|\mu| \neq \infty$ we obtain (98) with $r(\mu)$ and $u(\mu)$ depending on the matrix C. Note that as $|\mu| \to \infty$, then $r^2(\mu) + u^2(\mu) \to 0$. From (95) we have $|u(x,t)| = |\eta| e^{[p + r(\mu)]\mu^2 t}$. In order not to magnify $|\eta|$ for large μ and upon continuous dependence of the eigenvalues on parameter μ we conclude that p must be nonpositive. $\square$

Note: If $p = 0$, then $|u(x,t)| = |\eta| e^{r(\mu)\mu^2 t}$, which shows that matrix C is of importance (Example 4.5 below). If $p = 0$ and $C = 0$, then system (92a) will always admit bounded solutions for consistent η since then λ will be real and equal $(q + u(\mu))\mu^2$ (Example 4.6). Note also that there is a direct correspondence between our approach and the modal (Galerkin) DAEs. The plane wave solution relies on the pencil $(A, -B + C/\mu^2)$ (or equivalently on $(A, -\mu^2 B + C)$ if $s = i\lambda$ is used instead of $i\lambda/\mu^2$). In the modal analysis, if the boundary conditions implied a zero eigenvalue, then the pencil (A, C) was of interest, just like pencil (A, C) in the plane wave solution if $p = 0$.

Example 4.5 *Consider system (92a) with* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *Then* $det(As - B + C/\mu^2) = 0$ *gives* $s \equiv i\lambda/\mu^2 = -\frac{1}{4} + \frac{1}{4-\mu^2}$, *i.e.* $p = -\frac{1}{4}$, $r(\mu) = \frac{1}{4-\mu^2}$, $q = 0$, *and* $u(\mu) = 0$. *Since* p *is negative, therefore the system admits a plane wave solution. Note that* $B_1 - B_2 B_4^{-1} B_3 = -\frac{1}{4}$, *so* $p = -\frac{1}{4}$ *is an eigenvalue of* $B_1 - B_2 B_4^{-1} B_3$.

Example 4.6 *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \frac{1}{4}\begin{bmatrix} \theta & 0 \\ -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *for* $\theta \in \mathbf{R}$. *Then* $det(As - B + C/\mu^2) = 0$ *for* $s \equiv \frac{i\lambda}{\mu^2} = \frac{\theta}{4} - \frac{1}{\mu^2}$. *Since* $B_1 - B_2B_4^{-1}B_3 = \frac{\theta}{4}$ *we obtain* $p = \frac{\theta}{4}$, $q = 0$, $r(\mu) = -\frac{1}{\mu^2}$ *and* $u(\mu) = 0$. *If* $\theta = 0$, *then* $p = 0$, $\lambda = i$ *and* $|u(x,t)| = |\eta|e^{-t}$. *This solution is bounded, so the system admits a plane wave solution. Changing the* $(1,1)$ *entry of* C *to* -1 *yields* $\lambda = -i$ *and* $|u(x,t)| = |\eta|e^{t}$, *which is not acceptable for any* η.

Example 4.7 *Let* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ *and* $C = 0$. *Then* $\frac{i\lambda}{\mu^2} = \pm i$ *and* $p = 0$, $q = \pm 1$ *and* $t(\mu) = u(\mu) = 0$. *Therefore* $\lambda = \pm\mu^2$ *and (96) is admissible for* η *satisfying* $(\pm A\frac{i}{\mu^2} - B)\eta = 0$.

4.5 Equilibria and Their Stability for DAEs

Let us analyze what must to be taken into account when linearizing (1) around its equilibria. We restrict ourselves to time invariant DAE. $\overline{y}$ is an equilibrium of $F(y', y) = 0$ if and only if $F(0, \overline{y}) = 0$. If $F_y(0, \overline{y})$ is nonsingular, then $\overline{y}$ is isolated and can in principle be found by solving $F(0, \overline{y}) = 0$ by a numerical or symbolic method.

The invertibility of $F_y(0, \overline{y})$ is important in another way. The following lemma is used in the subsequent theorem.

Lemma 4.1 *Suppose that* A *and* B *is a regular pencil of matrices, that is,* $det(\lambda A + B)$ *is not identically zero. Then* $\hat{\lambda} = 0$ *is an eigenvalue of the pencil if and only if* $det(B) = 0$.

Proof. The eigenvalues of the pencil are those λ such that $det(\lambda A + B) = 0$. If 0 is an eigenvalue, then $det(B) = 0$. Conversely, we note that since the pencil is regular there are invertible matrices P, Q such that $PAQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}$, $PBQ = \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$ where N is a nilpotent matrix of index k. Thus $det(\lambda A + B)$ is a constant multiple of $det(\lambda I + D)det(\lambda N + I) = det(\lambda I + D)$. If D is singular, then $det(\lambda A + B) = 0$ for $\lambda = 0$. □

Theorem 4.1 *Suppose that* $\overline{y}$ *is an equilibrium of* $F(y', y) = 0$. *Suppose that the DAE satisfies (A1)-(A4) in Definition 1.4 in a neighborhood of* $(0, \overline{y}, 0)$. *Let* $A = F_{y'}(0, \overline{y})$, $B = F_y(0, \overline{y})$. *Suppose that* B *is nonsingular. Let* y *be* n *dimensional and* r *be the difference in rank of* $[G_{y'}, G_w]$ *and* $[G_{y'}, G_y, G_w]$ *for this system at* $(0, \overline{y}, 0)$. *Then*

1. *The local linearization* $A\tilde{y}' + B\tilde{y} = B\overline{y}$ *and the original DAE* $F(y', y) = 0$ *have the same dimensional solution manifold in a neighborhood of* $\overline{y}$.

2. $y' = \tilde{y}' + O(||y - \overline{y}||^2)$.

3. The dimension of the solution manifold is $n - r$.

Thus if the pencil $\lambda A + B$ *has* $n - r$ *finite eigenvalues with nonzero real part, then they will determine the stability properties of* $\overline{y}$ *on the solution manifold of* $F(y', y) = 0$.

Proof. Items 1 and 2 are from [8]. They imply that the eigenvalues of the pencil will determine the stability of the equilibrium provided the number of nonzero eigenvalues with nonzero real part is the same as the dimension of the solution manifold. Item 3 is from [9]. The final conclusion now follows. □

5 Travelling Waves, Conservation Laws and Nonlinear Differential Algebraic Equations

5.1 Preliminary Remarks

There has not been any substantial research reported on the application of DAE theory to the analysis of nonlinear infinite dimensional systems of conservation laws. One may expect enourmous difficulties associated with attempts to create a unified approach to this problem. Extending the results from linear to nonlinear systems usually requires some assumptions on the nature of the nonlinear problem, initial and/or boundary conditions, etc. Therefore in this section we will consider a particular nonlinear system, namely a nonlinear system of conservation laws and a special initial value problem, the Riemann problem. This problem is application oriented and the particular area of possible applications include gas dynamics, elastocity and magnetohydrodynamics [34], [35], [36], [37], [46], [47]. There is a close relation between travelling wave solutions in these areas and shock wave formation. However, we are not considering this relation in this thesis. Our traveling wave solutions in systems of conservation laws are smooth solutions. This will allow us to use the index theory of DAEs with smooth solutions as discussed earlier in section 3.1. Let us here only mention the simplest possible relation that exists between traveling and shock solutions for a relatively simple p-system (proof of the result can be found elsewhere [27], [46]).

Theorem 5.1 *The $p-$ system of conservation laws*

$$\begin{aligned} u_t - v_x &= 0 \\ v_t + p(u)_x &= 0 \end{aligned}$$

with $p' < 0$, $p'' > 0$ admits a Lax shock if and only if there is a travelling wave solution $\{u, v\} = \{f, g\}((x - st)/\mu)$ for the system

$$u_t - v_x = 0 \tag{101a}$$
$$v_t + p(u)_x = \mu u_{xx}. \tag{101b}$$

The number s is called the wave's speed.

In this chapter we analyze only continuous solutions of (101) and other similar systems introduced later on. Note that we have not yet discussed what is meant by the travelling wave solution of the nonlinear system. We mentioned the above results just to indicate how important the problem of finding travelling waves may be. The level of difficulty rises when more complicated systems are considered. Such examples include the system of 3 equations of conservation laws in gas dynamics [46], system of 7 equations in elastocity [35], and system of 7 equations in magnetohydrodynamics [47]. Another related example, although not a system of conservation laws, is a system of 5 nonlinear equations in hypoplasticity [37]. The problem of the existence of the travelling waves in all these areas is very important and mathematically nontrivial. In this section we will derive the DAE models of the travelling wave solutions in two important nonlinear systems of conservation laws. The first model describes the conservation laws in gas dynamics and the second model deals with magnetohydrodynamics. It is shown that the travelling wave solutions lead to an algebraic system of nonlinear equations for computation of the wave speed values and a DAE for computation of an orbit connecting the given two equilibria. Our derivation of DAEs is based on traveling wave solutions with constant speed s only. No other traveling waves (such as, for example, rarefaction waves) are considered in this thesis.

5.2 General System of Conservation Laws

We consider the following system of conservation laws

$$u_t + F(u)_x = [B(u)u_x]_x \tag{102}$$

where $u(x,t) \in \mathbf{R}^n$, $F(u) \in \mathbf{R}^n$ is smooth in a neighborhood N of $\mathbf{R}$ and the Jacobian $dF(u)$ has n real and disctinct eigenvalues in N. Matrix $B(u)$ is always real and in most cases singular with its entries depending on the viscosity coefficients, thermal conductivity, electric resistivity, etc. These coefficients, if included in the system of conservation laws, change the nature of system (102) from the hyperbolic ($B \equiv 0$) to parabolic ($B \neq 0$).

Definition 5.1 *We say that system (102) admits a travelling wave solution if there exists $s \in \mathbf{R}$ and u^r,$u^l \in \mathbf{R}^n$ such that $u(x,t) = u(x - st)$ for $(x,t) \in \mathbf{R}^n \times \mathbf{R}^+$, u satisfies (102), and $lim_{\psi\to-\infty}u(\psi) = u^l$, $lim_{\psi\to+\infty}u(\psi) = u^r$, and $lim_{\psi\to\pm\infty}u'(\psi) = 0$, where $\psi \equiv x - st$.*

In this thesis only smooth solutions of (102) are considered. The above definition of the admissibility of the travelling wave solution was used in the Theorem 5.1 above. The importance of this definition is now obvious, since it links the solution of a parabolic system of conservation laws to that of hyperbolic systems and its shock wave solution. The link is through the Riemann problem for the system $u_t + F(u)_x = 0$ for which the initial conditions $u(x,0) = u^l$ if $x < 0$, and $u(x,0) = u^r$ if $x > 0$, are assumed. Note that u^r and u^l are given equilibria for the solution $u(\psi)$. Therefore, the above problem is equivalent to the problem of connecting the two equilibria by a heteroclinic orbit.

In definition 5.1 and throughout this thesis we are interested in the traveling wave solutions with a constant speed s. As a result, the DAEs that are obtained are nonlinear autonomous DAEs. The wave solutions with time or space dependent wave speeds, such as rarefaction waves, are not considered in our analysis.

Although the definition 5.1 is formulated for a nonlinear system given by (102), one can consider analogous definitions for the linear form of (102), i.e. systems of the form

$$u_t + Bu_{xx} + Cu_x = 0 \tag{103}$$

Let us examine the admissibility of the travelling wave solution for system (103) with singular B. Introducing $u(x,t) = u(x - st)$ into (103) we obtain

$$su' + Bu'' + Cu' = 0 \tag{104}$$

Integrating (104) once from u^l to any u we get

$$s(u - u^l) + C(u - u^l) + Bu' = 0 \tag{105}$$

Since u^r is an equilibrium, therefore from (105) we have

$$s(u^r - u^l) + C(u^r - u^l) = 0 \tag{106}$$

Denoting $q \equiv u^r - u^l$ we obtain that $Eq = 0$, where $E = Is + C$. There exists a nontrivial solution of this equation if and only if the wave speed s is a finite eigenvalue of the pencil (I, C) and $q \in ker(Is + C)$. If we want to find an orbit connecting u^l with u^r, then we need to solve a DAE (105), which can be rewritten as

$$Bu' + Mu = Mu^l \tag{107}$$

where $detB = 0$, $M = Is + C$ (s is assumed to be known). (107) is linear and its particular solution is $u_p = u^l$. Now $Bu' + Mu = 0$ is a constant coefficient DAE. It has no solution with u^l and u^r as limits except $u = 0$. If $u^l = u^r$ then we get the trivial, constant solution $u(x - st) = u^l$. Hence there are no nontrivial traveling wave solutions in linear constant coefficient systems. We will show shortly that the same problem in nonlinear systems has different answer.

If $u = u(x - st)$ is to be a solution of (102), then substituting this solution into (102) we get

$$-su' + F(u)' = B(u)u'' \tag{108}$$

Integrating (108) once from the assumed equilibrium u^l to any state u, we obtain

$$-su + F(u) + su^l - F(u^l) = B(u)u' \tag{109}$$

Hence the nonlinear travelling wave DAE has the following form

$$B(u)u' + M(u) = M(u^l) \tag{110}$$

where $M(u) \equiv su - F(u)$ and the s is computed from the algebraic equation $s(u^r - u^l) - F(u^r) + F(u^l) = 0$.

Note that the later equation is the well-known *jump condition* (or *Rankine-Hugoniot condition*) for the existence of the shock solution in non-dissipative system $u_t + F(u)_x = 0$ [46]. Thus the wave speed s is determined by the *left* and *right* equilibria. A type of converse holds.

Theorem 5.2 *Fix u^l. Suppose that there exists a solution $\hat{u}$ of the DAE (110) which connects equilibrium u^l with the equilibrium u^r with wave speed $\hat{s}$. Suppose that the assumptions (A1)-(A4) in Definition 1.4 hold for (110) in a neighborhood of $\hat{u}$ and u^l, u^r and for s near $\hat{s}$.*

1. *If $-\hat{s} + F'(u^r)$ is nonsingular, then for s near $\hat{s}$ there will be a right equilibrium $u^r(s)$ and a solution of (110) connecting u^l to $u^r(s)$.*
2. *Suppose $[-\hat{s} + F'(u^r), u^r - u^l]$ is invertible when its ith column is deleted. let δ be the ith component of u^r. Then there will exist a new right equilibrium $\tilde{u}^r(\delta)$ with this same ith component and a solution connecting u^l and $\tilde{u}^r(\delta)$ for a wave speed $s(\delta)$ near $\hat{s}$.*

Proof. The proof is a straightforward application of the implicit function theorem and standard ODE theory, once we see that the assumptions (A1)-(A4) holding as s varies ensures that the solution manifold has fixed dimension and also varies smoothly with the chosen parameters [6], [10]. □

At an equilibrium $\overline{u}$, the linearization of (110) yields the pencil

$$B(\overline{u})\lambda + sI - F'(\overline{u}). \tag{111}$$

Note that equation (102) includes the form

$$u_t + F(u)_x = b(u)_{xx} \tag{112}$$

by letting $B(u) = b'(u)$. However, (102) is more general since not every $B(u)$ can be written as $b'(u)$ for some $b(u)$. The DAE (102) could also be written as a semi-explicit DAE by letting $v = u'$ but that would increase the index by one [1].

5.3 The p-system

Consider the travelling wave solutions $(u,v)(x,t) = (u,v)(x-st)$ of the *p-system* given in Theorem 5.1. Integrating the system along the travelling wave solution between (u^l, v^l) and a general state (u,v), then substituting $(u,v) = (u^r, v^r)$ one obtains

$$\begin{aligned} -s(u^r - u^l) - (v^r - v^l) &= 0 \\ -s(v^r - v^l) + p(u^r) - p(u^l) &= 0 \end{aligned}$$

This gives the possible wave speeds

$$s = \pm\sqrt{\frac{p(u^l) - p(u^r)}{u^r - u^l}} \tag{114}$$

If $u^r > u^l$ then we must have $p(u^r) < p(u^l)$. Otherwise, if $u^r < u^l$ then $p(u^r) > p(u^l)$. Equation (110) has the following form

$$\begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}' + \begin{bmatrix} su + v \\ sv - p(u) \end{bmatrix} = \begin{bmatrix} su^l + v^l \\ sv^l - p(u^l) \end{bmatrix} \tag{115}$$

The matrix pencil (111) gives that the eigenvalue at an equilibrium is $\lambda = p'(u) + s^2$. Note that $\lambda \neq 0$ precisely when $det(sI - F'(u)) \neq 0$ as promised by Theorem 5.2.

What is required in order to get the required traveling wave with two equilibria whose stability is determined by the linearization? One can transform (115) into a scalar equation in v only, i.e. $v' = f(v, s, \mu, v^l, u^l)$ which is solvable under the restriction imposed on p in Theorem 5.1. Then u is obtained from the first equation in (115). Since the solution manifold is one dimensional, in order to get a trajectory from u^l to u^r we must have u^l is unstable and u^r is stable. This can only happen if $p'(u^l) + s^2 > 0$ and $p'(u^r) + s^2 < 0$.

5.4 Gas Dynamics

The gas dynamics equations in the Eulerian coordinates take the following form [46]

$$\rho_t + (\rho w)_x = 0 \tag{116a}$$

$$(\rho w)_t + (p + \rho w^2)_x = 0 \tag{116b}$$

$$\{\rho(\frac{1}{2}w^2 + e)\}_t + \{\rho w(\frac{1}{2}w^2 + e) + pw\}_x = 0 \tag{116c}$$

where p (pressure) and e (energy) are some nonlinear functions of ρ (density) and T (temperature). The ρ, w (velocity), along with T are considered as dependent variables. It is easy to see that with $u = (\rho, w, T)^T \in \mathbf{R}^3$ one can transform system (116) into (102) with

$B(u) \equiv 0$. If we introduce the viscosity coefficient $\mu > 0$ and the thermal conductivity coefficient k>0 into system (116) then we obtain

$$\begin{aligned} h\rho_t + (\rho w)_x &= 0 & (117a) \\ (\rho w)_t + (p + \rho w^2)_x &= \mu w_{xx} & (117b) \\ \{\rho(\frac{1}{2}w^2 + e)\}_t + \{\rho w(\frac{1}{2}w^2 + e) + pw\}_x &= \mu(w w_x)_x + kT_{xx} & (117c) \end{aligned}$$

System (117) can be written in the general form (102) with $B = J^{-1}E$, and

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & \mu w & k \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 & 0 \\ w & \rho & 0 \\ \frac{1}{2}w^2 + e + \rho e_\rho & \rho w & \rho e_T \end{bmatrix} \tag{118}$$

Obvious calculations yield

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu/\rho & 0 \\ 0 & 0 & k/\rho e_T \end{bmatrix} \tag{119}$$

If $(\rho, w, T) \equiv [f((x - st)/\mu),\ g((x - st)/\mu), h((x - st)/\mu)]$, then we obtain a second order system of ODEs from (117). This system when integrated between some *left* state $u^l \equiv (f^l, g^l, h^l)'$ and genral state u gives

$$\begin{aligned} -s(f - f^l) + fg - (fg)^l &= 0 & (120a) \\ -s\{fg - (fg)^l\} + p + fg^2 - \{p + fg^2\}^l &= g' & (120b) \\ -s\{f(\frac{1}{2}g^2 + e) - \{f(\frac{1}{2}g^2 + e)\}^l\} + fg(\frac{1}{2}g^2 + e) & & \\ -\{fg(\frac{1}{2}g^2 + e)\}^l + pg - (pg)^l &= gg' + \frac{k}{\mu}h'. & (120c) \end{aligned}$$

or more compactly $B(u)u' + M(u)u = M(u^l)$ with

$$B(u) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g & \frac{k}{\mu} \end{bmatrix}. \tag{121}$$

Note that if we denote $u = (h, g)$, $v = f$, then (120) can be written in the form

$$\begin{aligned} F(u, u', v) &= 0 \\ G(u, v) &= 0, \end{aligned}$$

where $\partial F/\partial u' = \begin{bmatrix} k/\mu & g \\ 0 & 1 \end{bmatrix}$, and $\partial G/\partial v = -s + g$.

11pt Smilarly as in the case of the $p-system$ above, we have here the following system of three nonlinear algebraic equations for the computation of the wave speed ((120) with zero right hand side)

$$\begin{aligned} -s(f^r - f^l) + f^r g^r - f^l g^l &= 0 \qquad (123a)\\ -s(f^r g^r - f^l g^l) + p^r - p^l + f^r (g^2)^r - f^l (g^2)^l &= 0 \qquad (123b)\\ -s[f^r(\frac{1}{2}(g^2)^r + e^r) - f^l(\frac{1}{2}(g^2)^l + e^l)] + f^r g^r (\frac{1}{2}(g^2)^r + e^r) & \\ -f^l g^l (\frac{1}{2}(g^2)^l + e^l) + p^r g^r - p^l g^l &= 0 \qquad (123c) \end{aligned}$$

where e and p are functions of the descriptor vector $u \equiv (f, g, h)^T$.

The above system is used to compute the wave speeds. Having obtained the values of speed s we proceed to compute the orbits connecting u^r and u^l. To do this we need to solve the DAE

$$B(u)u' + M(u) = M(u^l) \qquad (124)$$

with $u(\infty) = u_r$, $B(u)$ given by (119) and

$$M(u) = \begin{bmatrix} sf - fg \\ sfg - p - fg^2 \\ sf(\frac{1}{2}g^2 + e) - fg(\frac{1}{2}g^2 + e) - pg \end{bmatrix} \qquad (125)$$

The Jacobian of this DAE depends on u, more precisely on one of its elements, namely g, but the rank of the Jacobian is constant and equal 2 if $k \neq 0$. If $k \neq 0$, then the system has index 1 provided that the gas velocity (represented by g) is not equal s (wave speed). The dimension of the solution manifold is 2. If $g = s$, then the system has higher index depending on functions p and e (i.e. on the type of gas). If $k = 0$ and $g \neq s$, then the index is still 1, but the dimension of the solution manifold is 2.

Example 5.1 *Consider system (117) with* $p = R\rho T$, $e = c_v T$, *and suppose we are looking for the travelling waves connecting the left state* u^l *with the right state* u^r *with the wave speed* $s = 1$. *Suppose also we know some components of* u^l *and* u^r, *as follows:* $\rho(-\infty) = 1$, $\rho(+\infty) = .5$, *and* $w(-\infty) = .5$. *Let* $R = c_v = 1$, *and* $k = \mu = 1$ *in (116). If we use (123) above to compute the remaining components of* u^l *and* u^r *then we obtain:* $w(+\infty) = 0$, $T(-\infty) = 0.3125$, *and* $T(+\infty) = 0.125$. *Thus, the two equilibria in (120) are:* $(f^l, g^l, h^l) = (1,\ 0.5,\ 0.3125)$ *and* $(f^r, g^r, h^r) = (0.5,\ 0,\ 0.125)$. *Solving system (124) with the initial conditions* $g(0) = g^l$ *and* $h(0) = h^l$ *and calculating* f *from the first equation in (120) we* *obtain the solutions shown in Fig.33. The 3D plot of $w(x,t)$ and the respective contour plot are given in Fig.34 and Fig.35, respectively.*

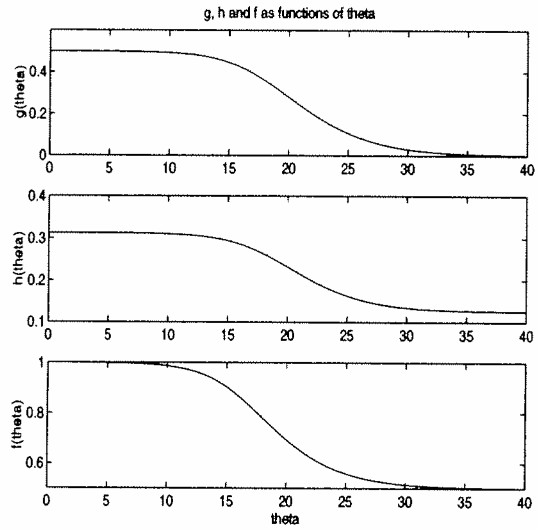

Fig.33. Traveling wave solutions.

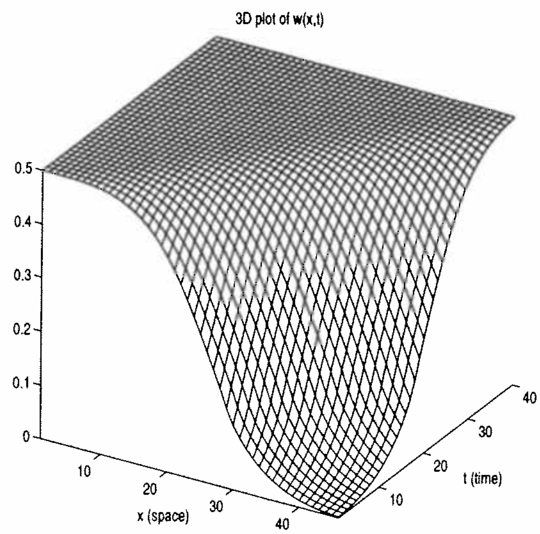

Fig.34. 3D solution for w.

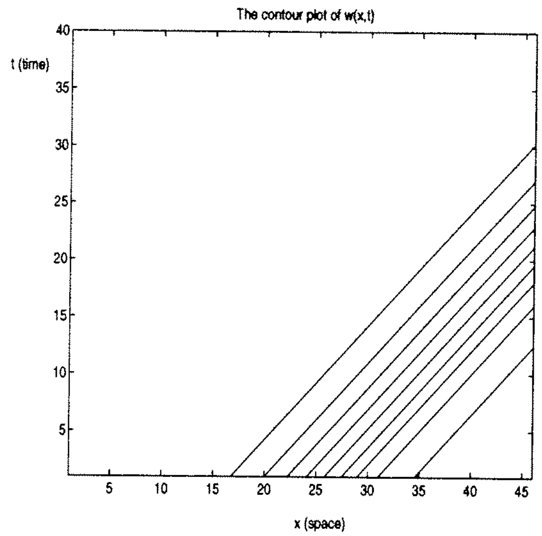

Fig.35. Contour lines for w.

The solution of the same problem but with $k = 0$ (i.e. one dimensional solution manifold) is shown in Fig.36 which presents the solution for $(f(\theta), g(\theta), h(\theta))$ between the *left* equilibrium (1, 0.5, -3.025) and the *right* equilibrium (5, -3.9, 4.235) with the wave speed $s = -5$.

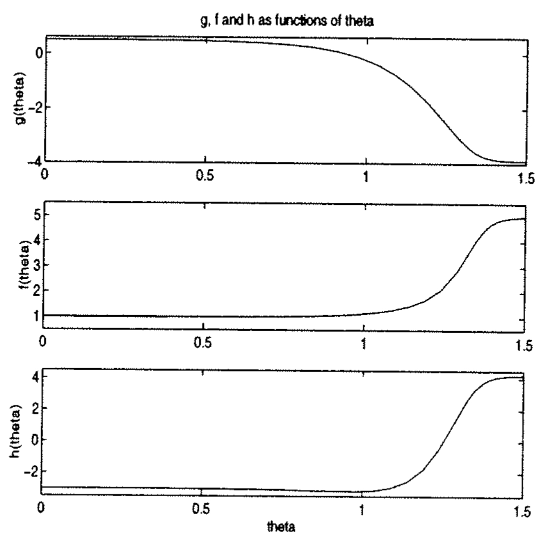

Fig.36. Traveling wave solutions ($k = 0$).

5.5 Magnetohydrodynamics

5.5.1 The MHD Equations

The MHD system of 7 nonlinear equations describes the flow of a conducting fluid in the presence of a magnetic field. The equations describe changes in the magnetic field, electric variables and the hydrodynamic variables. In general, the fluid dynamic equations are coupled with Maxwell's equations describing the electromagnetic effects in the system. The study of MHD has progressed first through hydrostatics, hydrodynamics, a study of steady flows of incompressible fluids, discussion of sound waves and stability of laminar flows, to the more modern problems of high speed compressible flow and turbulent motion.

The MHD problems can be placed both in theoretical and applied frameworks. The theoretical aspects include idealized problems in astrophysics and geophysics, such as discussions on magnetic storms, solar winds, cosmic magnetic fields, magnetic fields of sunspots, and analysis of the Earth's magnetopause during magnetic reconnection process [48],[23]. In all these cases the most important part of the work lies in formulating a mathematical model which describes the more important aspects of the problem [23].

The applied framework on the other hand emphasizes application and analysis of the MHD models in the fields of nuclear physics and engineering, as well as space research [24].

The dissipative MHD equations with resistivity, viscosity, and thermal conductivity have the following form [48]

$$\frac{\partial \rho}{\partial t} = \nabla.(\rho \mathbf{v}) \tag{126a}$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} = -\nabla.(\rho \mathbf{v}\mathbf{v} + \mathbf{I}(p + \frac{B^2}{2}) - \mathbf{B}\mathbf{B}) + \nu\nabla^2\mathbf{v} + (\mu + \frac{\nu}{3})\nabla(\nabla.\mathbf{v}) \tag{126b}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{126c}$$

$$\frac{\partial E}{\partial t} = -\nabla.((\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + p)\mathbf{v} + \mathbf{E} \times \mathbf{B}) + \nabla.\sigma.\mathbf{v} + \kappa\nabla^2(\frac{p}{\rho}) \tag{126d}$$

In the above, ρ ,p,$\mathbf{v}$, and $\mathbf{B}$ denotes the mass density, pressure, velocity, and magnetic field, respectively. γ is the ratio of the specific heats, the energy density E is given by $E = \frac{\rho v^2}{2} + \frac{B^2}{2} + p/(\gamma - 1)$, $v = ||\mathbf{v}||$, $B = ||\mathbf{B}||$ and the electric field is described by Ohm's law $\eta \mathbf{J} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, $\mathbf{J} = \nabla \times \mathbf{B}$, and $\eta =$ const is an electric resistivity. ν and μ are the two coefficients of viscosity, and κ is the thermal conductivity. Additionally, we have $\nabla.\mathbf{B} = 0$.

If we consider one-dimensional flow only, e.g. in x direction, then $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$, $\mathbf{B} = (B^x, B^y, B^z)'$ with $B^x = const$ and B^y, B^z are functions of x and t. Therefore we can rewrite (126) in the one-dimesnional case as follows [48]

$$\rho_t + (\rho u)_x = 0 \tag{127a}$$

$$
\begin{aligned}
(\rho u)_t + (\rho u^2 + P^*)_x &= (\mu + \frac{4}{3}\nu) u_{xx} && (127b)\\
(\rho v)_t + (\rho u v - B^x B^y)_x &= \nu v_{xx} && (127c)\\
(\rho w)_t + (\rho u w - B^x B^z)_x &= \nu w_{xx} && (127d)\\
B^y_t + (B^y u - B^x v)_x &= \eta B^y_{xx} && (127e)\\
B^z_t + (B^z u - B^x w)_x &= \eta B^z_{xx} && (127f)\\
E_t + [(E + P^*)u - B^x(B^x u + B^y v + B^z w)]_x &= (\mu + \frac{4}{3}\nu)(\frac{u^2}{2})_{xx} + \nu(\frac{v^2 + w^2}{2})_{xx} \\
&\quad + \eta \left(\frac{(B^y)^2 + (B^z)^2}{2} \right)_{xx} \\
&\quad + \kappa(\frac{p}{\rho})_{xx} && (127g)
\end{aligned}
$$

where $P^* = p + \frac{1}{2}||\mathbf{B}||^2$, $E = \frac{1}{2}\rho||\mathbf{v}||^2 + \frac{p}{\gamma-1} + \frac{1}{2}||\mathbf{B}||^2$, $\mathbf{v} = (u, v, w)^T$ is the velocity vector, $\mathbf{B} = (B^x, B^y, B^z)^T$ is the magnetic field, ρ, p, E, γ denote density, static pressure, energy and ratio of specific heats, respectively, and η, κ, and μ, ν are resistivity, thermal conductivity and two viscosity coefficients, respectively. The later four coefficients are constant.

If the system (127) admits traveling wave solutions

$$
u^T = (\rho, u, v, w, B^y, B^z, E) = (u_1, u_2, u_3, u_4, u_5, u_6, u_7)(x - st) \qquad (128)
$$

then we obtain the following DAE for the descriptor vector u

$$
\begin{aligned}
-s u_1' + (u_1 u_2)' &= 0 && (129a)\\
-s(u_1 u_2)' + (u_1 u_2^2 + P^*)' &= (\mu + \frac{4}{3}\nu) u_2'' && (129b)\\
-s(u_1 u_3)' + (u_1 u_2 u_3 - B^x u_5)' &= \nu u_3'' && (129c)\\
-s(u_1 u_4)' + (u_1 u_2 u_4 - B^x u_6)' &= \nu u_4'' && (129d)\\
-s u_5' + (u_2 u_5 - B^x u_3)' &= \eta u_5'' && (129e)\\
-s u_6' + (u_2 u_6 - B^x u_4)' &= \eta u_6'' && (129f)\\
-s u_7' + [(u_7 + P^*)u_2 - B^x(B^x u_2 + u_3 u_5 + u_4 u_6)]' &= (\mu + \frac{4}{3}\nu)(\frac{u_2}{2})'' \\
&\quad + \nu(\frac{u_3^2 + u_4^2}{2})'' + \eta(\frac{u_5^2 + u_6^2}{2})'' \\
&\quad + \kappa(\frac{p}{x_1})'' && (129g)
\end{aligned}
$$

If we integrate system (129) once with respect to $\psi \equiv x - st$, and use the fact that the derivative of u is zero at at the *left* state, then

$$
-s(u_1 - u_1^l) + u_1 u_2 - (u_1 u_2)^l = 0 \qquad (130a)
$$

$$
\begin{aligned}
-s[u_1u_2-(u_1u_2)^l]+u_1u_2^2+P^*-(u_1u_2^2+P^*)^l &= (\mu+\frac{4}{3}\nu)u_2' && (130b)\\
-s[u_1u_3-(u_1u_3)^l]+u_1u_2u_3-B^xu_5-(u_1u_2u_3-B^xu_5)^l &= \nu u_3' && (130c)\\
-s[u_1u_4-(u_1u_4)^l]+u_1u_2u_4-B^xu_6-(u_1u_2u_4-B^xu_6)^l &= \nu u_4' && (130d)\\
-s(u_5-u_5^l)+u_2u_5-B^xu_3-(u_2u_5-B^xu_3)^l &= \eta u_5' && (130e)\\
-s(u_6-u_6^l)+u_2u_6-B^xu_4-(u_2u_6-B^xu_4)^l &= \eta u_6' && (130f)\\
-s(u_7-u_7^l)+(u_7+P^*)u_2-B^x(B^xu_2+u_3u_5+u_4u_6) & && \\
-[(u_7+P^*)u_2-B^x(B^xu_2+u_3u_5+u_4u_6)]^l &= (\mu+\frac{4}{3}\nu)(\frac{u_2^2}{2})' && \\
&\quad +\nu(\frac{u_3^2+u_4^2}{2})'+\eta(\frac{u_5^2+u_6^2}{2})' && \\
&\quad +\kappa(\frac{p}{u_1})' && (130g)
\end{aligned}
$$

where $B^x = const$, and the static pressure is generally a function of the components of u, i.e. $p = p(u)$.

Thus (130) is a system of 7 nonlinear DAEs of the form $G(u)u' + M(u) = M(u^l)$ with the singular Jacobian $G(u)$, as follows

$$
G(u) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mu+\frac{4}{3}\nu & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \nu & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \nu & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \eta & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \eta & 0\\
\alpha_1(u) & \alpha_2(u) & \alpha_3(u) & \alpha_4(u) & \alpha_5(u) & \alpha_6(u) & \alpha_7(u)
\end{bmatrix} \qquad (131)
$$

with

$$
\begin{aligned}
\alpha_1(u) &= \kappa\frac{p_{u_1}u_1-p}{u_1^2} && (132a)\\
\alpha_2(u) &= (\mu+\frac{4}{3}\nu)u_2+\kappa\frac{p_{u_2}}{u_1} && (132b)\\
\alpha_3(u) &= \nu u_3+\kappa\frac{p_{u_3}}{u_1} && (132c)\\
\alpha_4(u) &= \nu u_4+\kappa\frac{p_{u_4}}{u_1} && (132d)\\
\alpha_5(u) &= \eta u_5+\kappa\frac{p_{u_5}}{u_1} && (132e)\\
\alpha_6(u) &= \eta u_6+\kappa\frac{p_{u_6}}{u_1} && (132f)\\
\alpha_7(u) &= \kappa\frac{p_{u_7}}{u_1} && (132g)
\end{aligned}
$$

where p_{u_i} is the derivative of $p(u)$ with respect to u_i, i=1,...,7.

5.5.2 A Few Comments on the MHD DAE

The properties of a traveling wave solution of a nonlinear system of conservation laws in magnetohydrodynamics given above are here summarized from the point of view of differetial algebriac equations. We consider several particular cases of the traveling wave MHD DAEs. The nonlinear system of 7 PDEs describing the relations between electromagentic and hydrodynamic parts is considered in the one-dimensional setting with x denoting the spatial and t the time variables. The dependent variables are density ρ, 3 components of the velocity vector $(u, v, w)^T$, 2 components of the magnetic induction vector $(B^y, B^z)^T$ and energy E. All those variables are functions of x and t. The B^x component of the magnetic field is constant due to the requirement that $\nabla B = 0$. Also, the special, so-called planar case of the above system consists of only 5 equations since B^z and w are equal zero.

The MHD DAEs are closely related to the Riemann problem for the PDE system (127) with zero right hand side. The Riemann problem is an initial value problem in which we are looking for a solution of $u_t + f(u)_x = 0$ between two states, the so-called *left* and *right* states. That is, the initial condition is: $u(x,0) = u^l$ for $x < 0$ and $u(x,0) = u^r$ for $x \geq 0$. It turns out that under mild conditions this problem is equivalent to finding the traveling wave solution in variable $\theta = x - st$ with s denoting the wave speed for system (127) with non-zero right hand side. Historically, the first approach to the Riemann problem was based on discontinuity theory in which the problem is solved provided certain jump conditions (the Rankine-Hugoniot conditions) were satisfied for the hyperbolic system of nonlinear first-order equations [46],[47]. Next, the continuous approach was used in which the dissipative terms were taken into account. The limit of the soulution of such dissipative system is called the *viscosity* solution of (130). Such a limit is a weak-star limit in L^∞ and u is a piecewise C^1 solution containing a single shock. The convergence is uniform off of any neighborhood containing the shock. The idea behind adding the viscosity terms is simple. All real physical systems always have some kind of dissipative mechanism that is modeled by second order derivatives, like u_{xx}. This is particularly true in magnetohydrodynamic and gas dynamics problems, where any fluid or gas has at least "small", but nonzero, electrical resistivity or a nonzero thermal conductivity. These issues are only mentioned here. The main focus of our research is on the relation between DAEs and traveling wave solutions rather than the analysis of shock solutions. Our goal is to use the DAE theory to study properties of traveling waves and related topics such as singularities of the MHD DAEs, bifurcations in DAEs and their structures (semi-explicit and conservative DAEs).

The system of conservation laws with viscosity coefficients becomes a parabolic one and the corresponding finite dimensional DAE (i.e. (130) with the independent variable $\theta = x - st$) has the *left* and *right* states as its two equilibria.

Note that the viscosity approach uses the second-order system (due to the dissipative mechanism represented by constants η, μ, ν, and κ in (127)), but the final system (say, travelling wave system) (130) is a nonlinear DAE.

There are several reasons to believe that one can benefit from using the DAE approach for analyzing the travelling wave MHD system. First, the MHD DAEs come from the parabolic (dissipative) PDEs and parabolic systems are usually easier to solve than the hyperbolic ones (used by the discontinuity method). Therefore one may prefer working with parabolic systems rather than hyperbolic. Second, one may also benefit from using the DAE approach when trying to explain the physical properties that stand behind the results. The DAE theory has positive results regarding stability analysis of equilibria on the constraint manifolds [39],[43], bifurcation analysis [41], [42], numerical methods [1], [2], [28], [38], etc. We shall show that one can link several particular results known in the theory of magnetohydrodynamics with such notions of DAEs like *index*, *impasse points*, *singular points*, etc. There are cases in the MHD (see eq.(148) below) where the whole MHD system can be described just by one DAE system with all the necessary equilibria, instead of two ODE subsystems, each on a separate branch of the constraint manifold (the so-called *subsonic* and *supersonic* branches). Third, and perhaps the most important reason for using the DAE approach is due to the recent results which show that the discontinuity theory (i.e. MHD system without dissipative terms) can not identify all possible shock solutions that may occur. Instead, it was shown in [47],[48], mainly by several numerical experiments, that *intermediate shocks* can physically exist if small but positive constants η, μ, ν, and κ are taken into consideration. It was long believed that the *intermediate* shocks were nonphysical, since certain jump conditions in the discontinuity theory were not satisfied. The numerical examples with the dissipative systems like (127) show that this belief was not correct. This gives one more reason why the use of MHD DAE (130) to analyse the shock structure may be superior to the discontinuity approach. Therefore, it is our belief that in the future research on the application of DAEs to systems of conservation laws, one may develop the DAE based tools to determine whether certain types of shock solutions are or are not acceptable. The formation and structure analysis of the *intermediate* shocks in MHD rely, at least at the current stage of research, mostly on numerical rather than analytical results [48]. Our only intention is to point out the possibilities that may exist when one links DAEs and shock solutions in systems of conservation laws together. The more detailed analysis is beyond the scope of this thesis.

Since our DAE approach to systems of conservation laws is based on the viscosity method (i.e. the continuous approach) and since the viscosity method seems, as described above and urgued elsewhere [47],[48], to have advantage over the discontinuous approach, it may happen in the future that the DAE theory becomes a powerful tool that can be used in many analytical proofs in the large area of MHD problems. Finally, in the numerical examples presented in this chapter we have used the dissipative coefficients, such as μ, ν, η, κ and other constants, such as γ and B^x found in the existing literature on this topic, e.g. [23], [24],[48].

5.5.3 Basic Properties of the MHD DAE

Let us first analyze the existence and number of possible equilibria in the MHD DAE (130). We consider (130) with the zero right hand side, and we assume that all components of the *left* state and the value of s are given. Note that as described above the *left* state is an equilibrium of the system. Then the following result holds true.

Proposition 5.1 *If the wave speed and all components of the left state are known, then the MHD DAE (130) can admit* **at most** *4 different equilibria (including the left state).*

Proof. One can show (e.g. by using symbolic calculations with MAPLE) that the 7 algebraic equations in (130) can be transformed into a *4th* degree polynomial equation in one of the components of the *right* state (e.g. u_6^r). Such a polynomial equation has **at most** 4 real solutions (e.g. 4 different values of u_6^r), and the rest of the components of the *right* state are functions of the particular component involved in the polynomial equation. A MAPLE code used to prove this is given in Appendix I. □

Proposition 5.2 *The real roots of the polynomial*

$$p(u_6) = (au_6^3 + bu_6^2 + cu_6 + d)(u_6 - u_6^l) \tag{133}$$

with

$$\begin{aligned}
a &= (B^x)^2[(u_6^l)^2 + (u_6^l)^2] && (134a)\\
b &= u_6^l[(u_6)^2 + (u_6^l)^2][u_1^l(2-\gamma)(s-u_2^l)^2 + (B^x)^2(\gamma-1)] && (134b)\\
c &= -[(B^x)^2 - (u_2^l)^2 u_1^l]c^* && (134c)\\
c^* &= [(u_5^l)^2 + (u_6^l)^2](2-\gamma)\gamma + u_1^l(1-\gamma)\left(\gamma[(u_2^l)^2 + (u_3^l)^2 + (u_4^l)^2] - (u_2^l - s)^2\right)\\
&\quad -(B^x)^2(\gamma^2+1) - 2\gamma u_7^l(1-\gamma) && (134d)\\
d &= -(u_6^l)^3(\gamma+1)[(u_2^l - s)^2 u_1^l - (B^x)^2]^2 && (134e)
\end{aligned}$$

are the values of u_6 at the equilibria of of the traveling wave MHD DAE. Other components of u are functions of u_6.

Proof. Factor out common terms in the coefficients of the polynomial equation in Proposition 5.1. □

Note how various components of the *left* state and parameters γ, B^x, and s influence coefficients a, b, c and d as well as the number of equilibria of the system. The u_6^l is a component of the *left* equilibrium. Several special cases follow. If we assume that

$$u_2^l - s = \pm\sqrt{\frac{(B^x)^2}{u_1^l}} \tag{135}$$

then $c = d = 0$ and $p(u_6) = u_6^l(au_6 + b)(u_6 - u_6^l)$. We have three roots: u_6^l , $-u_6^l$ $(= -b/a)$, and 0 (double root). Variable u_2 has the meaning of velocity and the right hand side in the last formula is known as the speed of the Alfven wave [46].

If $u_6^l = 0$, then $b = d = 0$ and we have double root $u_6 = 0$ and two roots $u_6 = \pm\sqrt{\frac{-c}{a}}$, provided c/a is negative. On the other hand, if $u_5^l = u_6^l = 0$ (two components of the magnetic firld are zero), then $a = b = d = 0$ and the only roots are 0 (single) and u_6^l.

As mentioned above one of the most important issues in the Riemann problem is the possibility of connecting of various equilibria via travelling waves. The equilibria of the DAE (132) are the *left* and *right* states in the Riemann problem. Therefore the components of u^l and u^r in (132) are exactly the same as in the initial condition: $u(x, 0) = u^l$ for $x < 0$ and $u(x, 0) = x^r$ for $x > 0$ for system (127).

Although the system (132) can admit 4 equilibria, not all possible pairs can be regarded as the equilibria of the Riemann problem. This is due to the fact that some of the equilibria may lie on different parts of the constraint manifold and no smooth connection between equilibria can be made. In general, there are two reason for that. First, the constraint manifold may consist of disjoint branches with no common points. Second, even if the constraint manifold is a smooth surface it may consists of two branches connected at points which cannot be reached by a continuous solution. The differential system becomes singular at such points. This holds true both in planar and non-planar cases. There are many possibilities that exist and one can analyze only several special cases. This is due to the many parameters involved here (at least 7 components of the *left* equilibrium, coefficient γ, parameters B^x and s). Also, different dissipative mechanisms yield different behaviors, depending on which of the 4 coefficients η, μ, ν and κ are non-zero. Dynamic behavior of the system may even change within the same dissipative mechanism if different values of the same dissipative coefficients are chosen. Therefore it is rather difficult to formulate and prove general statements regarding the connectibility of equilibria. Part (b) of this section gives an insight into this problem based on analysis of several numerical examples. However, before doing that, perhaps a slightly different approach to the Riemann problem is worth mentioning.

Suppose we know all 7 components of the *left* state and one of the components of the *right* state. We also assume that B^x and γ are known, but s is unknown. Then the following result holds true.

Proposition 5.3 *If all 7 components of the left state and one component of the right state are known, then the MHD DAE (130) admits* **at most** *6 different value of s, the wave speed.*

Proof. In the same manner as in the proof of Proposition 5.2 one can use symbolic calculations and MAPLE to transform 7 algebraic equations (i.e. (130) with zero right hand side) to a 6 degree polynomial equation in variable s. The roots of this polynomial are the 6 wave speeds, and the other unknown components of the *right* state are computed as functions of s. □

5.5.4 Dissipative Mechanism and Existence of the Travelling Waves

The dissipative mechanism represented by coefficients η, μ, ν, and κ in (130) gives different structures for the DAEs depending on which of those coefficients are zero and which are non-zero. Below, some typical cases are considered. For a particular problem it is usually possible to eliminate some variables. This is not necessary for a general analysis. Also the simplification can be complex. However, we shall do so to simplify the presentation. The result will sometimes be a DAE or an ODE depending on the problem. The most interesting from the DAE point of view, are DAEs in semi-explicit and conservative forms, either in planar and non-planar cases. Some typical cases are described below. Specific numerical examples will be examined more carefully in the next subsection.

Case 1: μ only.
If only μ is non-zero, then from (130) one obtains a scalar equation u_2 in the form

$$p(u_2)u_2' = q(u_2), \tag{136}$$

where $p(u_2)$ and $q(u_2)$ are polynomials in u_2 of degree 3 and 4, respectively, If we let $p(u_2) = h_{u_2}(u_2)$, then we have a DAE in conservation form

$$\frac{d}{dt}[h(u_2(t))] = q(u_2(t)) \tag{137}$$

The system (137) can be rewritten as a semi-explicit DAE

$$\begin{aligned} y' &= q(u_2) && (138a)\\ 0 &= y - h(u_2) && (138b) \end{aligned}$$

Note that numerically solving (137) is not the same as numerically solving (138). For instance, a backward Euler gives different solutions. Transforming (137) to (138) increases the index by one. However, it is possible for some DAEs in conservation form that such transformation does not change the index.

Theorem 5.3 *Suppose that the DAE*

$$
\begin{aligned}
h'(u)u' &= f_1(u,t) & (139a)\\
0 &= f_2(u,t) & (139b)
\end{aligned}
$$

is an undex ν DAE. Suppose also that the derivative array that determines u' can be computed by ν differentiations of (139b) and $\nu - 1$ differentiations of (139a). Then the index of

$$
\begin{aligned}
y' &= f_1(u,t) & (140a)\\
0 &= f_2(u,t) & (140b)\\
0 &= y - h(u) & (140c)
\end{aligned}
$$

is also ν.

Proof. With one differentation of (140c) combined with (140a) we get (139a) and (140a) is (139b). Thus the equations gotten by ν differentations of (139b) and $\nu - 1$ differentations of (139a) can also be gotten by ν differentations of (140b), $\nu - 1$ differentations of (140a) and $\nu - 1 + 1 = \nu$ differentations of (140c). Thus the index of (140) is also ν. □

Note that the four possible roots of $q(u_2)$ are the equilibria of the system. Roots of $p(u_2)$ are the *singular* points of the system. They play an important role in the singularity induced bifurcation in DAEs, which is discussed in Section 5.6.

Case 2: η only.
First, let's analyze this case in a most general setting. The DAE (130) has the form

$$
\begin{aligned}
-su_1 + u_1u_2 - c_1 &= 0 & (141a)\\
-su_1u_2 + u_1u_2^2 + P^* - c_2 &= 0 & (141b)\\
-su_1u_3 + u_1u_2u_3 - B^*u_5 - c_3 &= 0 & (141c)\\
-su_1u_4 + u_1u_2u_4 - B^*u_6 - c_4 &= 0 & (141d)\\
-su_5 + u_2u_5 - B^xu_3 - c_5 &= \eta u_5' & (141e)\\
-su_6 + u_2u_6 - B^*u_4 - c_6 &= \eta u_6' & (141f)\\
-su_7 + (u_7 + P^*)u_2 - B^x(B^xu_2 + u_3u_5 + u_4u_6) - c_7 &= \eta(u_5u_5' + u_6u_6') & (141g)
\end{aligned}
$$

where c_i are constants determined by u_l. That is, c_i is the *ith* entry of the vector $u_l - F(u_l)$ in (109).

The Jacobians of (141) are

$$F_{u'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 & \eta u_5 & \eta u_6 & 0 \end{bmatrix} \tag{142}$$

$$F_u = - \begin{bmatrix} u_2 - s & u_1 & 0 & 0 & 0 & 0 & 0 \\ (u_2 - s)u_2 + P_1^* & (2u_2 - s)u_1 + P_2^* & P_3^* & P_4^* & P_5^* & P_6^* & P_7^* \\ (u_2 - s)u_3 & u_1 u_3 & (u_2 - s)u_1 & 0 & -B^x & 0 & 0 \\ (u_2 - s)u_4 & u_1 u_4 & 0 & (u_2 - s)u_1 & 0 & -B^x & 0 \\ 0 & u_5 & -B^x & 0u_2 - s & 0 & 0 & \\ 0 & u_6 & 0 & -B^x & 0 & u_2 - s & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \end{bmatrix} \tag{143}$$

where $P_i^* = \frac{\partial F^*}{\partial u_i}$ and α_i are nonzero entries. Let

$$\Theta = - \begin{bmatrix} -s + u_2 & u_1 & 0 & 0 & 0 \\ -su_2 + u_2^2 + P_1^* & -su_1 + 2u_1u_2 + P_2^* & P_3^* & P_4^* & P_7^* \\ -su_3 + u_2u_3 & u_1u_3 & -su_1 + u_1u_2 & 0 & 0 \\ -su_4 + u_2u_4 & u_1u_4 & 0 & -su_1 + u_1u_2 & 0 \\ \alpha_1 & \alpha_2 - u_5^2 - u_6^2 & \alpha_3 + B^x u_5 & \alpha_4 + B^x u_6 & \alpha_7 \end{bmatrix} \tag{144}$$

To simplify the remaining discussion we rewrite (141) as

$$\begin{aligned} f_1(u_1, u_2) &= 0 && \text{(145a)} \\ f_2(u_1, u_2, u_3, u_4, u_5, u_6, u_7) &= 0 && \text{(145b)} \\ f_3(u_1, u_2, u_3, u_5) &= 0 && \text{(145c)} \\ f_4(u_1, u_2, u_4, u_6) &= 0 && \text{(145d)} \\ f_5(u_2, u_3, u_5) &= \eta u_5' && \text{(145e)} \\ f_6(u_2, u_4, u_6) &= \eta u_5' && \text{(145f)} \\ f_7(u_1, u_2, u_3, u_4, u_5, u_6, u_7) &= \eta(u_5 u_5' + u_6 u_6') && \text{(145g)} \end{aligned}$$

Proposition 5.4 *For the system (141) with Θ defined by (144) we have that*

1. *If $det(\Theta) \neq 0$ and $u_1 \neq 0$, then the DAE (141) is an index one DAE with a 2 dimensional solution manifold.*

2. *If the pencil* $\{F_{u'}, F_u\}$ *has 2 nonzero eigenvalues with nonzero real parts at an equilibrium, then these eigenvalues determine the stability properties of that eqilibrium on the solution manifold.*

3. *The solution manifold is given by (141a)–(141d) and* $\hat{f}_7 = -u_5 f_5 - u_6 f_6 + f_7 = 0$.

Proof. Adding $-u_5$ times row 5 to row 7 and $-u_6$ times row 6 to row 7 converts the pencil $\{F_{u'}, F_u\}$ to the pencil

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \Theta_{11} & * & \Theta_{12} \\ * & * & * \\ \Theta_{21} & * & \Theta_{22} \end{bmatrix} \tag{146}$$

where $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$. The pencil $\{F_{u'}, F_u\}$ is index one if and only if Θ is nonsingular. If $F_{u'}$ has constant rank, then the DAE is index one if and only if the pencil of its linearization is also index one. This direct relationship between the index of the pencil and the index of the DAE is not true for higher index DAEs. □

The solution manifold given by $f_1 = 0, ..., f_4 = 0, \hat{f}_7 = 0$ contains all the equilibrium points. Provided its Jacobian is full row rank we get a well defined manifold.

There are several problems which can now arise in order for there to exists a traveling wave that connects two equilibria. First, the solution manifold may consists of more than one component. In order to connect them the equilibria need to be on the same component. Secondly, even if the equilibria both lie on the same component of the manifold, there can be difficulties. The equation $det(\Theta) = 0$ defines another, possibly empty, manifold which we call the *singularity* manifold. In some problems the singularity manifold may intersect the solution manifold. If the equilibria are on opposite sides of the singularity surface, it may not be possible to connect them with a solution. The specific examples given in the next section illustrate these problems.

Some further simplifications of system (141) are possible. For example, by using symbolic calculations in Maple one is able to transform (141) to the form

$$u_5' = f(u_5, u_2) \tag{147a}$$
$$u_6' = g(u_6, u_2) \tag{147b}$$
$$0 = h(u_5, u_6, u_2) \tag{147c}$$

where h is quadratic in all 3 variables.

This system has several interesting properties. First of all, one is able to write (147) in the form

$$J(u, u')u' = H(u) \tag{148}$$

where $u = (u_5, u_6)'$.

However, the Jacobian $J(u, u')$ has variable structure and depends not only on u but also on u'. This makes the analysis of such a system rather difficult, and none of the existing numerical algorithms for solving DAEs can be applied here if (148) has index greater than 1. Instead, the usual approach was to break (141) into two subsystems, each valid on a separate branch of the constraint manifold $h = 0$. These two branches are called the *supersonic* and *subsonic* and their structures may have various forms. In the non-planar case considered in example 2 below, the manifold $h = 0$ consists of two separate surfaces in variables u_2, u_5 and u_6. All 4 equilibria in this example lie on an "egg"-like part of the constraint manifold. The 3 equilibria lie on a *subsonic* branch, while the 4th equilibrium is on a *supersonic* branch. No smooth connection exists between those two branches. However, it is possible to have traveling wave solution between some of the 3 equilibria on the *subsonic* branch.

The common points between the *subsonic* and *supersonic* branches of the constraint manifolds are called the *sonic* points where the wave speed equals the *sound* speed. One can prove the following property linking the *sonic* points with the singular points of the DAE system.

Proposition 5.5 *The MHD DAE singular points (i.e. common points of $h = 0$ and $h_{u_2} = 0$) are the sonic points of the system.*

Proof. Consider first the planar case. Note that with a usual meaning of the varibales $u_5 \equiv B^y$, $u_2 \equiv v$, $u_6 = 0$, the planar MHD DAE the planar case has the following form

$$\frac{dB^y}{d\theta} = f(B^y, v) \tag{149a}$$

$$0 = h(B^y, v) \tag{149b}$$

This semiexplicit DAE has singularity if $\partial h/\partial v = 0$. Since $\partial h/\partial v = (\partial h/\partial B^y)(\partial B^y/\partial v)$, therefore at the singularity $\partial B^y/\partial v = 0$. On the other hand, it is known [48] that for any wave speed s we have

$$\frac{\partial \rho}{\partial B^y} = \frac{B^y}{s^2 - a^2} \tag{150a}$$

$$\frac{\partial(\rho v)}{\partial B^y} = \frac{(v - s)B^y}{s^2 - a^2} \tag{150b}$$

where a is the *sound* speed ($a^2 = \partial p/\partial \rho$). Note that ρ denotes density ($= u_1$) and therefore it is a finite quantity. The last two formulas yield

$$\rho \frac{\partial B^y}{\partial v} = \frac{s^2 - a^2 \rho}{s B^y} \tag{151}$$

If $s = a$ then $\partial B^y/\partial v = 0$. This is exactly the case where the MHD DAE is singular. □

Case 3: η and μ only.

If only η and μ are non-zero in the dissipative mechanism in system (130), then the system can be written in the form

$$z'(\theta) = f(z(\theta)) \tag{152a}$$

$$y(\theta) = g(z(\theta)) \tag{152b}$$

where $z \equiv (u_2, u_5, u_6)'$, $y \equiv (u_1, u_3, u_4, u_7)$, and f, g are nonlinear functions. Now, we solve (152a) explicitly for u_2, u_5 and u_6 , then use (152b) to find the remaining variables. Instead of having two separate constraint manifolds, or separate branches of the same manifold we have here a simple nonlinear ODE with 2, 3 or 4 equilibria. Parameters η and μ influence the qualitative behavior near the equilibria and therefore the possiblility of connecting those equilibria with each other may depend on the ratio μ/η.

Case 4: η and κ only.

If η and κ are the only non-zero dissipative coefficients in (130) then in the non-planar case we have the system

$$\begin{aligned}\kappa\eta h(u_2,u_5,u_6)u_2' &= f(u_2,u_5,u_6) \quad &(153a)\\ \eta u_5' &= g_1(u_2,u_5,u_6) \quad &(153b)\\ \eta u_6' &= g_2(u_2,u_5,u_6) \quad &(153c)\end{aligned}$$

where $h(u_2,u_5,u_6)=0$ defines the singularity manifold.

System (153) is a conservative DAE, which can be transformed into a semi-explicit DAE, but this usually increases the index by 1.

Case 5: κ and μ only.

If κ and μ are non-zero, then one has a system in the form

$$\begin{aligned}h(u_2)w(u_2)u_2' &= f(u_2,u_7) \quad &(154a)\\ w(u_2)u_7' &= g(u_2,u_7) \quad &(154b)\end{aligned}$$

where $h(u_2)$ and $w(u_2)$ are polynomials in u_2. Numerical example 4 covers this case.

Note that other combinations of the dissipative parameters are possible, but the cases above are interesting enough for further numerical analysis.

5.6 Numerical examples.

5.6.1 Example 1: μ only

Consider (130) with the right hand side equal 0 and the *left* equilibrium (denoted by A hereafter) $u^l = [1,2,0.25,0.75,1,0.5,0.25]^T$ and $s=1$, $B^x=0.2$, $\gamma=1.4$. Solving the nonlinear system of 7 algebraic equations one obtains the following 3 solutions of (130) with zero right hand side (i.e. the potential *right* equilibria (denoted by D,B and C, respectively)

$$u^r = \begin{bmatrix} -53.75952301\\ 0.9813986445\\ -3.226374724\\ -0.9881873619\\ -16.38187362\\ -8.190936809\\ -582.1136634 \end{bmatrix}, \begin{bmatrix} 2.104461466\\ 1.475180951\\ 0.4911957824\\ 0.8705978912\\ 2.205978912\\ 1.102989456\\ -1.033924030 \end{bmatrix}, \begin{bmatrix} -1.343641280\\ 0.2557537381\\ -0.1948210585\\ 0.5275894707\\ -1.224105293\\ -0.6120526464\\ 1.574389398 \end{bmatrix} \quad (155)$$

Suppose that we are interested in finding the travelling wave solutions of (130) with $\eta=\nu=\kappa=0, \mu=1$. This means that only the second and seventh equations in (130) have non-zero right hand side equal u_2' and u_2u_2', respectively. The solution manifold has dimension 1. In fact one can obtain an autonomous DAE $u_2' = F(u_2)$ as follows

$$u_2' = \frac{750u_2^4 - 3534.25u_2^3 + 5625.515u_2^2 - 3391.7288u_2 + 555.3976}{625u_2^3 - 1925u_2^2 + 1976u_2 - 676} \tag{156}$$

It can be easily checked that the numerator of the right hand side of (156) has zeros at the 4 equilibria given above, i.e. 2, 0.981398644, 1.475180951, 0.2557537381.

We can not connect all of them with orbits corresponding to the travelling waves. Because of the zero $u_2 = 1$ in the denominator of (156), we can connect 2 with 1.475180951 and 0.981398644 with 0.2557537381, only. Note that we can connect the equilibria 0.2557537381 and 0.981398644 only if 0.2557537381 is the *left* equilibrium and 0.981398644 is the *right* equilibrium. Using the linearization of (156) we compute that u^l is unstable, B, D are stable, and C is unstable. Fig.37 shows the solution of (130) for the six states between the *left* equilibrium above and the second equilibrium in (155), which serves as the *right* equilibrium.

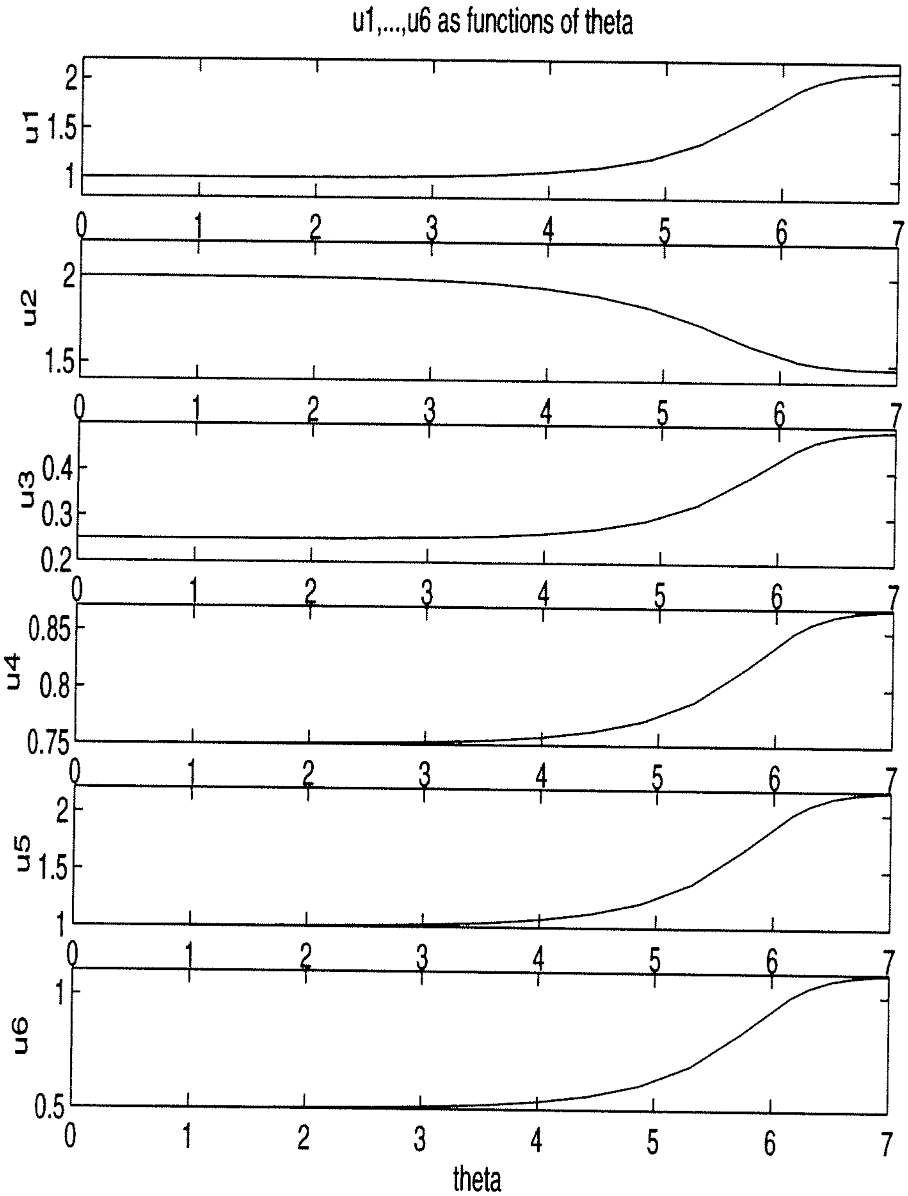

Fig.37. Solutions $u_1(\theta), ..., u_6(\theta)$.

5.6.2 Example 2: η only

Consider now a different set of constants on the right hand side of (130), namely $\mu = \nu = \kappa = 0$, $\eta \neq 0$. The *left* equilibrium is chosen to be as before. Now, the fifth, sixth and seventh equations in (130) have the right hand side non-zero, but the solution manifold has dimension 2 and the system has the same four equilibria given above. By doing the symbolic calculations with MAPLE and eliminating u_1, u_3, u_4 and u_7 from (130) we obtain the following 3 equations involving u_2, u_5, u_6 and their derivatives ($s = 1$, $B^x = 0.2$, $\gamma = 1.4$)

$$
\begin{aligned}
\eta u_5' &= u_5 u_2 - 1.04 u_5 - 0.96 && \text{(157a)}\\
\eta u_6' &= u_6 u_2 - 1.04 u_6 - 0.48 && \text{(157b)}\\
\eta u_5 u_5' + \eta u_6 u_6' &= 0.73 u_5^2 + 0.73 u_6^2 - 2.8315 - 0.75 u_5^2 u_2 \\
&\quad -0.75 u_6^2 u_2 + 7.897 u_2 - 3 u_2^2 && \text{(157c)}
\end{aligned}
$$

It is easy to check that all four equilibria A,B,C and D satisfy the above equations. That is, the right-hand side of (157) is zero for the respective values of u_2, u_5 and u_6 in u^l and (155). Note that u_2' is not present in (157), but (157c) is quadratic in u_2. Therefore, if one computes the u_2 from (157c), then the two solutions for u_2 are obtained. Pluging any of those solutions into the first two equations in (157) and transforming them into explicit form yields several systems of 2 nonlinear equations in u_5 and u_6. This happens because of several quadratic terms involved. Some of those 'new' systems may not have all the required equilibria. However, a careful analysis of those systems shows that one is able to derive 2 explicit systems in u_5, u_6 such that one of them has equilibria A,B and D, and the second has only one equilibrium, namely equilibrium C. The phase flow of the first system is shown in Fig.38. We are not able to get just one explicit system in u_5 and u_6 with all 4 equilibria.

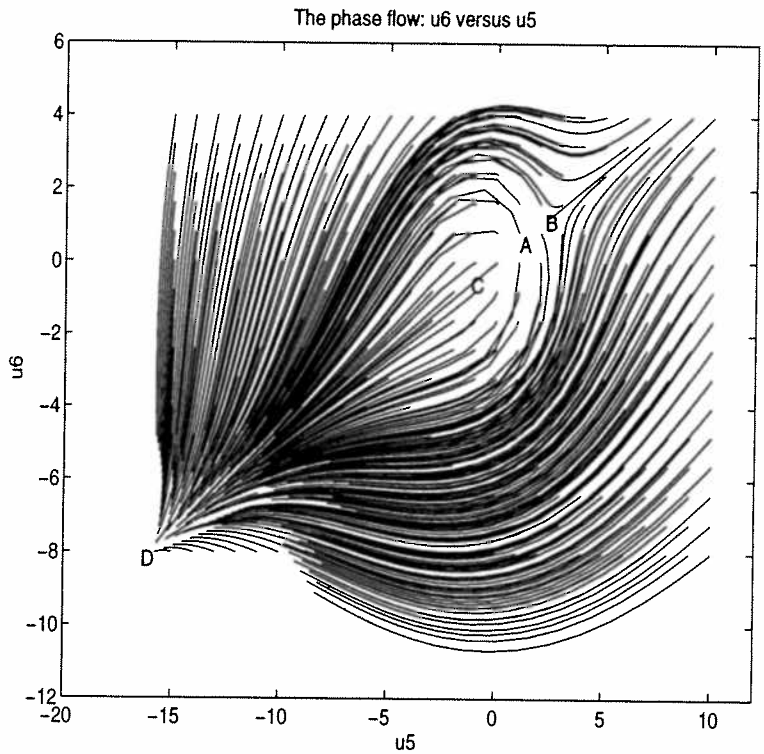

Fig.38. The phase flow on the subsonic branch of the constraint manifold.

We are however, able to get an implicit system in u_5 and u_6 with all 4 equilibria. Let us compute u_2 from (157a) and plug it into the second and third equations. This process gives

$$\eta u_5 u_6' - \eta u_6 u_5' = -0.48u_5 + 0.96u_6 \tag{158a}$$

$$\begin{aligned} 1.75\eta u_5^3 u_5' + \eta u_6 u_5^2 u_6' + 0.75\eta u_6^2 u_5 u_5' & \\ -1.657\eta u_5 u_5' + 3\eta^2 (u_5')^2 + 5.76\eta u_5' &= -0.05u_6^2 u_5^2 - 0.05u_5^4 + 2.13658u_5^2 - 0.72u_6^2 u_5 \\ & \quad -0.72u_5^3 + 1.59072u_5 - 2.7648 \end{aligned} \tag{158b}$$

One can check that (158) has all four equilibria A,B,C and D, but the system is in implicit form and its Jacobian depends not only on u_5, u_6 but also on u_5'. Note that it follows from (157) that for any equilibrium we have $u_5 = 2u_6$ and then from (158b) one gets the equilibria being the zeros of the $4-th$ degree polynomial of the right-hand side of (158b).

Similarly, if we compute u_2 from (157b) and plug it into (157a), (157c) then we obtain

$$\eta u_6 u_5' - \eta u_5 u_6' = 0.48u_5 - 0.96u_6 \tag{159a}$$

$$\begin{aligned}\eta u_5 u_6^2 u_5' + 1.75\eta u_6^3 u_6' + 0.75\eta u_5^2 u_6 u_6' & \\ +3\eta^2(u_6')^2 - 1.657\eta u_6 u_6' + 2.88\eta u_6' &= -0.05u_5^2u_6^2 - 0.05u_6^4 - 0.36u_6^3 + 2.13658u_6^2 \\ & \quad -0.36u_5^2u_6 + 0.79536u_6 - 0.69120 \qquad (159b)\end{aligned}$$

It can be checked that the above system has all 4 desired equilibria A,B,C and D, but as before, the system is an implicit one and its Jacobian depends on u_5, u_6 and u_6'.

Note that both (158) and (159) are in the general form

$$F(u', u) = 0 \tag{160}$$

and after linearization we obtain the following eigenvalues at the equilibria

$$\begin{aligned} eig(A) &= \{0.2316; 0.4800\} && (161a) \\ eig(B) &= \{-0.0939; 0.2176\} && (161b) \\ eig(C) &= \{-0.9570; -0.3921\} && (161c) \\ eig(D) &= \{-0.0160; -0.0293\} && (161d) \end{aligned}$$

Thus equilibrium A is a repelling node, B is a saddle, and both C and D are attracting nodes.

5.6.3 Example 3: η only (planar case)

Let the *left* state be

$$u^l = [u_1^l, 0.5, 0.2, -1, 10]^T \tag{162}$$

with $B^x = 2$, $\gamma = 1.4$ and $s = 1$. We have chosen 3 different values of u_1^l, namely 15, 5 and 1. In the case $u_1^l = 15$ we have 4 equilibria, all of them lying on the closed branch of the constrained manifold near the origin. Three of these equilibria are on the lower part of this branch (i.e. below the singularity manifold) and one is above the singularity manifold. As in the non-planar case the two branches are called *subsonic* and *supersonic* branches, respectively. There are two singularity points on the constrained manifold where the constrained manifold $h = 0$ crosses the higher-index manifold $h_{u_2} = 0$. Any solution joining the equilibrium on the *supersonic* branch with an equilibrium on the *subsonic* branch must go through a singularity point. The stability analysis of the four equilibria shows that equilibria 2 and 4 are unstable and 3 is stable (*subsonic* branch). Therefore the only connection between equilibria on the *subsonic* branch are $2 \rightarrow 3$ and $4 \rightarrow 3$. The equilibrium on the *supersonic* branch is unstable. If $u_1^l = 5$ we have two unstable equilibria lying on separate branches of the constrained manifold. In the case when $u_1^l = 1$ we have 2 separate branches and 2 equilibria, namely $\mathbf{u}^l$ and $(0.122, -3.093, 3.878, -1.920, 4.853)$.

5.6.4 Example 4: κ and η only (planar case)

We present here a planar case of κ and η only. In the planar case one assumes that $u_4 = u_6 = 0$ in (130). Thus we have a DAE in u_2 and u_5 only.
Let

$$u^l = [8, 0.1, 2, -0.1, 10]^T, \; B^x = 2, \; \gamma = 1.4, \; s = 1, \; \kappa = 1, \; \eta = 0.1 \qquad (163)$$

Then we obtain four equilibria. Three of them are in the *subsonic* and one in the *supersonic* region. Both regions are separated by the singularity manifold: $h(u_2, u_5) = \kappa\eta(1.5463 - 2u_2 + 0.06944u_5^2) = 0$. The phase portrait is shown in Fig.39.

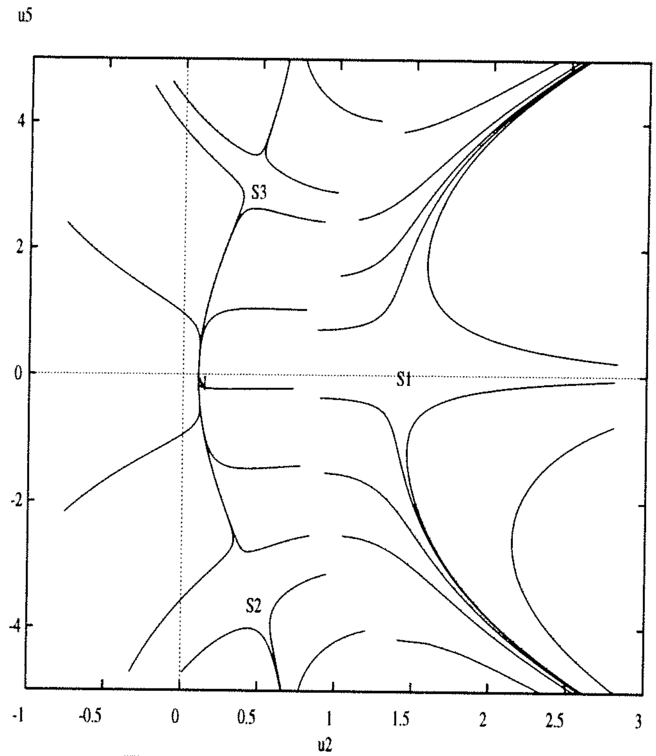

Fig.39. The phase portrait.

5.6.5 Example 5: η only (disjoint constraint manifold)

If we consider the non-zero η only and the planar case with

$$u^l = [1, 0.5, 0.2, -1, 10]^T, \; B^x = 2, \; \gamma = 1.4, \; s = 1 \qquad (164)$$

then we obtain two equilibria only. The constraint manifold consists of two disjoint branches as shown in Fig.40.

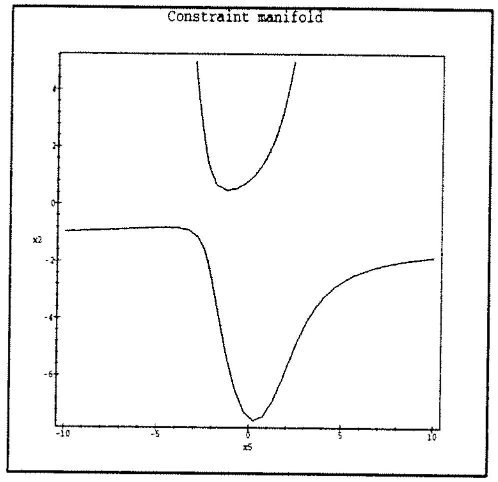

Fig.40. Disjoint branches of a constraint manifold.

The two equilibria u^l and $[0.122, -3.093, 3.878, -1.920, 4.853]$ lie on disjoint branches and no smooth solution exists between them.

5.7 Singularity Induced Bifurcation in MHD

As shown in the previous section the existence of traveling waves in MHD DAEs is restricted by the presence of singularities which do not allow for smooth connections of equilibria on different sides of the singularity. The singularity may be in the form of a 3D surface [40], or a 2D curve (see Subsection 5.4 with κ and μ only), or just a point (Subsection 5.6.1 with μ only). In any case the constraint manifold is divided into separate branches. This is due to the quadratic equation describing such manifold. In other cases the singularity manifold may not intersect the constraint manifold at all, but no traveling wave solution exists due to the equilibria lying on disjoint branches of the constraint manifold (see Subsection 5.5.3 above).

An interesting case is when an equilibrium is at the singularity. If a DAE depends on a parameter, such as one of the components of the left equilibrium, then it may happen that by changing this parameter we are able to shift an equilibrium to the singularity. This problem is known in the DAE literature as a singularity induced bifurcation and has been applied in the last few years only for the analysis of an electric power system [29], [49]. Below we analyze this bifurcation in the context of semi-explicit DAEs. The following theorem from [29] is the basis for such an analysis.

Theorem 5.4 *Consider a parameter dependent DAE*

$$u' = f(u,v,p) \tag{165a}$$
$$0 = g(u,v,p) \tag{165b}$$

with $f : \mathbf{R}^{n+m+q} \to \mathbf{R}^n$, $g : \mathbf{R}^{n+m+q} \to \mathbf{R}^m$, $u \in U \subset \mathbf{R}^n$, $v \in V \subset \mathbf{R}^m$, $p \in P \subset \mathbf{R}^q$.

If $q = 1$, $\Delta(u,v,p) \equiv det\left[\frac{\partial g(u,v,p)}{\partial v}\right]$, *and*

1. $f(0,0,p_0) = 0$, $g(0,0,p_0) = 0$, $\frac{\partial g}{\partial v}$ *has a simple zero and* $trace\left[\frac{\partial f}{\partial v} adj(\frac{\partial g}{\partial v})\frac{\partial g}{\partial u}\right] \neq 0$,

2. $\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}$ *is nonsingular,*

3. $\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial p} \\ \frac{\partial \Delta}{\partial u} & \frac{\partial \Delta}{\partial v} & \frac{\partial \Delta}{\partial p} \end{bmatrix}$ *is nonsingular,*

then there exists a smooth curve of equilibria in $\mathbf{R}^{n+m+1}$ *which passe s through* $(0,0,p_0)$ *and is transversal to the singular surface at* $(0,0,p_0)$. *When* p *increases through* p_0 *one eigenvalue of the system, (i.e. an eigenvalue of*

$$J = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}(\frac{\partial g}{\partial v})^{-1}\frac{\partial g}{\partial u} \tag{166}$$

evaluated along the equilibrium locus) moves from $\mathbf{C}^-$ *to* $\mathbf{C}^+$ *if* $b/c > 0$ *(respectively, from* $\mathbf{C}^+$ *to* $\mathbf{C}^-$ *if* $b/c < 0$*) along the real axis by diverging through* ∞. *The other* $n-1$ *eigenvalues remain bounded and stay away from the origin. The constants* b *and* c *can be computed by evaluating*

$$b = -trace\left[\frac{\partial f}{\partial v} adj(\frac{\partial g}{\partial v})\frac{\partial g}{\partial u}\right] \tag{167a}$$
$$c = \frac{\partial \Delta}{\partial p} - \left[\frac{\partial \Delta}{\partial u}, \frac{\partial \Delta}{\partial v}\right]\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}^{-1}\begin{bmatrix} \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial p} \end{bmatrix} \tag{167b}$$

Proof. See [29]. □

Note that when an equilibrium is placed at the singularity then one of its eigenvalues changes from either $-\infty$ to $+\infty$ (or from $+\infty$ to $-\infty$). That is the solution reaches and leaves the equilibrium with an infinite speed. In the particular examples below we will be able to prove that trajectories reach and leave an equilibrium at the singularity in a finite time. The situation is similar to that in a simple DAE: $u' = v$, $0 = v^2 - u$. Note that $u \neq 0$, except at $t = 0$ (we assume that $u = 0$ is reached at $t = 0$, initial time is $t_0 < 0$ and $u(t_0)$ is finite). It remains to show that t_0 is finite. This is in fact the case, since $u(t)$ changes according to: $u' = \pm\sqrt{u}$, which has a solution $u(t) = (t-c)^2/4$ and the initial condition is $u(t_0) < \infty$. Then $c = t_0 \pm 2\sqrt{u(t_0)}$ and $u(t) = [t - t_0 \pm 2\sqrt{u(t_0)}]^2/4$. For $t = 0$ we have $u(0) \equiv 0 = [-t_0 \pm 2\sqrt{u(t_0)}]^2/4$, therefore $t_0 = \pm 2\sqrt{u(t_0)}$ which is a finite quantity. Note that the $-$ sign is used when the trajectory approaches $u = 0$ at $t = 0$ and the $+$ sign is used when the trajectory leaves $u = 0$ at $t = 0$.

As a consequence of Theorem 5.4 we have the following corollary. Since we will use the technique later we include a proof.

Corollary 5.1 *If a DAE satisfies the conditions given in Theorem 5.4, then there always exists a trajectory to and from the equilibrium placed at the singularity.*

Proof. Suppose p is such that the equilibrium is at the singularity. Solutions of the DAE (165) are included in those of

$$\begin{aligned} u' &= f(u, v, p) && (168a) \\ 0 &= g_u' + g_v v' && (168b) \end{aligned}$$

or equivalently

$$\begin{aligned} u' &= f(u, v, p) && (169a) \\ v' &= -(g_v)^{-1} g_u f && (169b) \end{aligned}$$

The trajectories of (169) away from $\Delta = det(g_y) = 0$ are also trajectories of

$$\begin{aligned} u' &= \Delta f(u, v, p) && (170a) \\ v' &= -\Delta (g_v)^{-1} g_u f && (170b) \end{aligned}$$

away from $\Delta = 0$. The assumptions give us a nontrivial trajectory of (170) which goes to and from the equilibrium on the singularity surface $\Delta = 0$. This in turn gives us a trajectory of (169) which goes toward the singularity. However, $g = c$ is an invariant of (169). Since $g = 0$ by construction at the equilibrium we must have $g = 0$ along the trajectory. But then this trajectory is a solution of (168). $\square$

In the two examples below we will show how to apply theorem 5.4 in the case of $n = 2$, $m = p = 1$ and $n = m = 1$, respectively. Both examples perfectly identify the divergence of an eigenvalue of (166) at the bifurcation point. Behavior described in the Singularity Induced Bifurcation Theorem 5.4 is understandable intuitively. The drop by 1 in the rank of $\frac{\partial g}{\partial v}$ results in the divergence of an eigenvalue through $\pm\infty$.

5.7.1 Singularity Induced Bifurcation with η only

Consider the traveling wave MHD DAE with non-zero η only. Such a system can be reduced to the following form ??

$$\begin{aligned} u_5' &= f_1(u_5, u_2, p) && \text{(171a)} \\ u_6' &= f_2(u_6, u_2, p) && \text{(171b)} \\ 0 &= g(u_5, u_6, u_2, p) && \text{(171c)} \end{aligned}$$

where g is quadratic in u_2. Let the left state be

$$u^l = [p, 0.5, 0.2, 0.2, 1, 1, 10]^T,\ B^x = 2,\ \gamma = 1.4,\ s = 1 \tag{172}$$

where p is the parameter (it represents density at the left equilibrium). By solving the system of four nonlinear equations, namely

$$\begin{aligned} f_1(u_5, u_2, p) &= 0 && \text{(173a)} \\ f_2(u_6, u_2, p) &= 0 && \text{(173b)} \\ g(u_5, u_6, u_2, p) &= 0 && \text{(173c)} \\ g_{u_2}(u_5, u_6, u_2, p) &= 0 && \text{(173d)} \end{aligned}$$

one is able to place any of the existing four equilibria at the singularity.

Suppose we find a p such that the singularity induced bifurcation theorem holds and an equilibrium is placed at the singularity. Then there are trajectories going in and out of the equilibrium. We wish to see whether they do so in finite time. Using equation (141a) we get that (141c)– (141f) become

$$\begin{aligned} c_1 u_3 - B^x u_5 - c_2 &= 0 && \text{(174a)} \\ c_1 u_4 - B^x u_6 - c_4 &= 0 && \text{(174b)} \\ -s u_5 + u_2 u_5 - B^x u_3 - c_5 &= \eta u_5' && \text{(174c)} \\ -s u_6 + u_2 u_6 - B^x u_4 - c_6 &= \eta u_6' && \text{(174d)} \end{aligned}$$

Suppose that $c_1 \neq 0$. Then we can solve (174a) and (174b) to get

$$\begin{aligned} -s u_5 + u_2 u_5 - B^x c_1^{-1}[B^x u_5 + c_2] - c_5 &= \eta u_5' && \text{(175a)} \\ -s u_6 + u_2 u_6 - B^x c_1^{-1}[B^x u_6 + c_4] - c_6 &= \eta u_6' && \text{(175b)} \end{aligned}$$

This leads to the following result.

Lemma 5.1 *Suppose that $c_1 \neq 0$. Then there exist constants α_1, α_2, not both zero, such that $\alpha_1 u_5 + \alpha_2 u_6 = 0$ is invariant under solutions of (141).*

Proof. Let $z = \alpha_1 u_5 + \alpha_2 u_6$. Then from (175) we have that $z' = [-s + u_2 - c_1^{-1}(B^x)^2]z + \phi$ where $\phi = -\alpha_1(c_1^{-1}c_3 B^x + c_5) - \alpha_2(c_1^{-1}c_4 B^x + c_6)$. Take α_1, α_2 so that ϕ is zero. Then z satisfies a linear homogeneous differential equation

$$z' = [-s + u_2 - c_1^{-1}(B^x)^2]z \tag{176}$$

If $z(t)$ is zero for some t_0, then it is zero for all t_0. □

Now suppose that at the equilibrium the value of u_2 is such that $[-s+u_2-c_1^{-1}(B^x)^2] \neq 0$. Suppose that the trajectories given by the singularity induced bifurcation theorem do not reach or leave the origin in finite time. Then we can conclude by integrating either to the singularity, or backward in time to it, depending on the sign of $[-s + u_2 - c_1^{-1}(B^x)^2]$, that the singularity lies on $z = 0$. We focus now on the trajectories on $z = 0$. On this curve we can solve for one of u_5, u_6 in terms of the other. We assume it is u_5. Then we get that (174) reduces to a system in u_5, u_2. We translate the equilibrium to the origin. Keeping the same name for our new variables and using the fact that (171c) is quadratic in u_5 and u_6 with no products of them, we have

$$\begin{aligned} u_5' &= \alpha u_5 + \beta u_2 + u_5 u_2 && (177a) \\ 0 &= u_2^2 + (a + bu_5 + cu_5^2)u_2 + du_5 + eu_5^2 && (177b) \end{aligned}$$

The additional requirement that the origin is at the singularity gives $a = 0$. We are considering the case where the origin lies on a manifold defined by (177b) with $a = 0$. Thus (177b) must have real solutions for u_2 if u_5 is near zero. Thus

$$(bu_5 + cu_5^2)^2 - 4(du_5 + eu_5^2) \geq 0 \tag{178}$$

for u_5 near zero. Assume $d \neq 0$. The largest term is $-4du_5$. Thus we must have $-4du_5 > 0$. Then u_5 does not change sign near the origin. Let $u_5 = \kappa v^2$ where κ is either 1 or -1 depending on the sign of u_5. Then we have

$$\begin{aligned} 2\kappa v v' &= \kappa\alpha v^2 + \beta u_2 + \kappa v^2 u_2 && (179a) \\ 0 &= u_2^2 + (b\kappa v^2 + cv^4)u_2 + |d|v^2 + ev^4 && (179b) \end{aligned}$$

From (179b) we then get that $u_2 = 2\sqrt{|d|}v + o(v)$ and (179a) becomes $2v' \approx \alpha v + \kappa\beta\sqrt{|d|} + vu_2$. But $v \to 0$ and u_2 is bounded near the equilibrium. Thus along the trajectory, near the equilibrium (origin), we have that v' is bounded away from zero. Thus the v trajectories leave and arrive in finite time. Hence the same holds for u_5. Therefore we have the following theorem.

Theorem 5.5 *Suppose that only $\eta \neq 0$. Suppose that the equilibrium u_e is placed at the singularity by chosing u_2^l. Let u^l be such that $c_1 \neq 0$ in (141). Suppose that u_5^e and u_6^e are nonzero and $d \neq 0$ in (177). Then there exist solutions of (141) which reach and leave u^e in finte time.*

The following example is chosen for illustration. For the particular choice of $\mathbf{u}^l$ as in (172) the traveling wave MHD DAE is

$$
\begin{aligned}
u_5' &= -u_5 + 0.5 + u_5 u_2 + 8(u_5 - 1)/p && \text{(180a)}\\
u_6' &= -u_6 + 0.5 + u_6 u_2 + 8(u_6 - 1)/p && \text{(180b)}\\
0 &= 1.5u_2^2 p - 2.356u_2 p + 13.3u_2 - 1.75u_5^2 u_2 - 1.75u_6^2 u_2 + 1.75u_5^2 \\
&\quad +1.75u_6^2 - 0.5u_5 - 0.5u_6 + 0.803p - 7.4 + (8u_5 + 8u_6 - 4u_5^2 - 4u_6^2 - 8)/p && \text{(180c)}
\end{aligned}
$$

One can check that all conditions in Theorem 5.4 are satisfied.

Solving (173) we obtain the bifurcation parameter $p_0 = 17.97829718$ together with the following equilibrium at the singularity: $(u_2, u_5, u_6) = (0.6082, -1.0346, -1.0346)$. Note that in general solution of (173) is not unique. The other possible solutions for p_0 are: $p_0 = \{3.941980273, 11.44859813, 183, 7727649\}$. Each of these p_0's corresponds to different equilibrium being moved to the singularity. Table 1 illustrates location of the equilibria and values of eigenvalues when p changes between 17.50 and 19.00. The constraint manifold for $p = p_0$ is shown in Fig.41. Both the constraint and singularity manifolds for $p = p_0$ are shown in Fig.42.

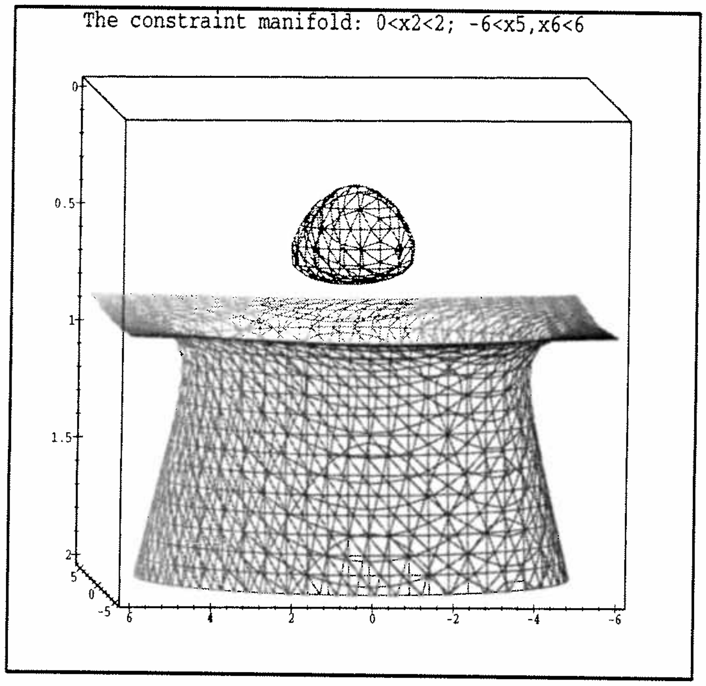

Fig.41. The constraint manifold for $p = p_0$

Note that (176) with $\alpha_1 = 1$, $\alpha_2 = -1$ takes the form

$$z' = (7 - u_2)z \tag{181}$$

where $z = u_5 - u_6$ and $u_2 \to 0.6082$. All of the conditions are met and there are trajectories reaching and leaving the equilibrium in finite time.

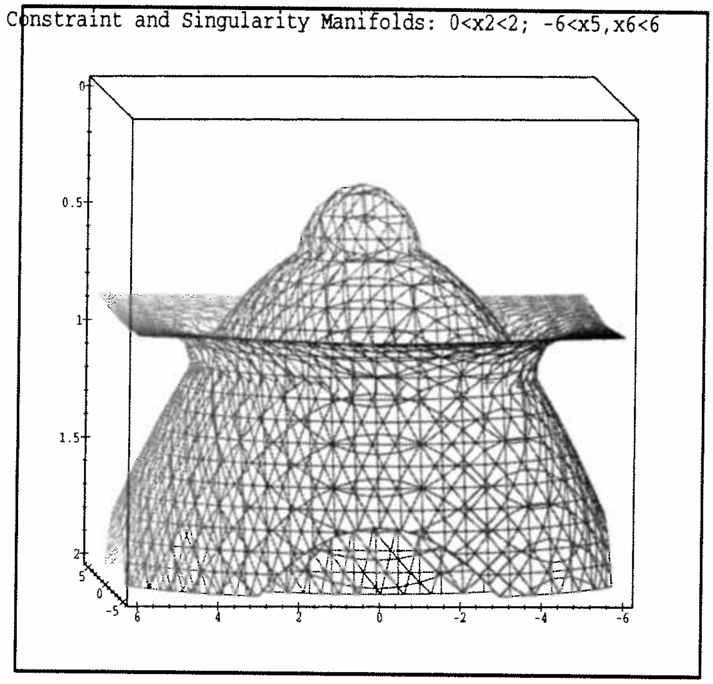

Fig.42. The constraint and singularity manifolds for $p = p_0$.

Note that the left state does not move with the change of p. For $p < p_0$ the three equilibria lie on the *supersonic* branch of the constraint manifold, whereas for $p > p_0$ we have two equilibria on each (*subsonic* and *supersonic*) branch.

5.7.2 Singularity Induced Bifurcation with κ and η only

Consider now the case when κ and η are non-zero (Subsection 5.6.4). The DAE has the following structure in the planar case

$$\kappa\eta h(u_2, u_5, p)u_2' = f(u_2, u_5, p) \tag{182a}$$

$$\eta u_5' = g(u_2, u_5, p) \tag{182b}$$

where $h(u_2, u_5, p) = 0$ defines the singularity manifold and h, f and g are all polynomial functions.

We can place an equiulibrium at the singularity by solving the system

$$h(u_2, u_5, p) = 0 \tag{183a}$$

$$f(u_2, u_5, p) = 0 \tag{183b}$$

$$g(u_2, u_5, p) = 0 \tag{183c}$$

Note that placing an equilibrium at the singularity means that we approach a constant state (zero time derivative), but on the other hand the dynamics are fast due to the singularity.

Let the planar MHD system have the following left state

$$u^l = [p, 0.5, 0.2, 1, 10]^T, \; B^x = 2, \; \gamma = 1.4, \; s = 1 \tag{184}$$

and $\kappa = 1$, $\eta = 1$. We have the following system

$$\begin{aligned}
(250pu_2 - 202p - 125u_5^2 + 875)u_2' &= -0.125(-20000u_5^2u_2^2p + 1500u_2^2p^3 \\
&\quad -160000u_5^2u_2 + 40000u_5^2u_2p - 2328u_2p^3 \\
&\quad +160000u_2u_5 - 1750u_5^2u_2p^2 + 12250u_2p^2 \\
&\quad -10000u_2u_5p + 160000u_5^2 + 18000u_5p \\
&\quad -24000u_5^2p - 400p + 1750u_5^2p^2 \\
&\quad -160000u_5 + 789p^2 - 500u_5p^2) && \text{(185a)} \\
u_5' &= -10u_5 + 10u_2u_5 + 5 + 80(u_5 - 1)/p && \text{(185b)}
\end{aligned}$$

For such a system one of the possible bifurcation values of p is $p_0 = 18.72228883$ for which a saddle is placed at the singularity. The coordinates of that saddle at the singularity are

$$(u_2, u_5) = (0.6467800031, -0.9814200462) \tag{186}$$

The phase portrait shown in Fig.42 (and zoomed in in Fig.43) indicates a possible connection between a saddle and a node lying in separate regions divided by the singularity curve $h = 0$. The connection is via an intermediate equilibrium (saddle) placed at the singularity. The stable manifold of that saddle is not changed during the bifurcation process, but the unstable manifold is changed so that divergence of the corresponding eigenvalue through $\pm\infty$ occurs. In this particular example there exist two traveling wave solutions: one going through the singularity from saddle S_1 to node N and the other from saddle S_3 to node N without crossing the singularity.

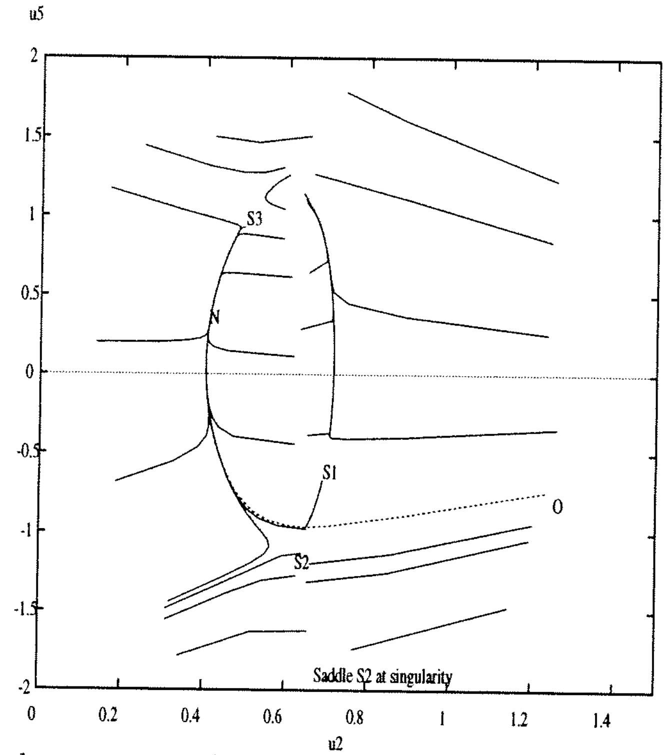

Fig.43. The phase portrait of system (182) with saddle S_2 at singularity.

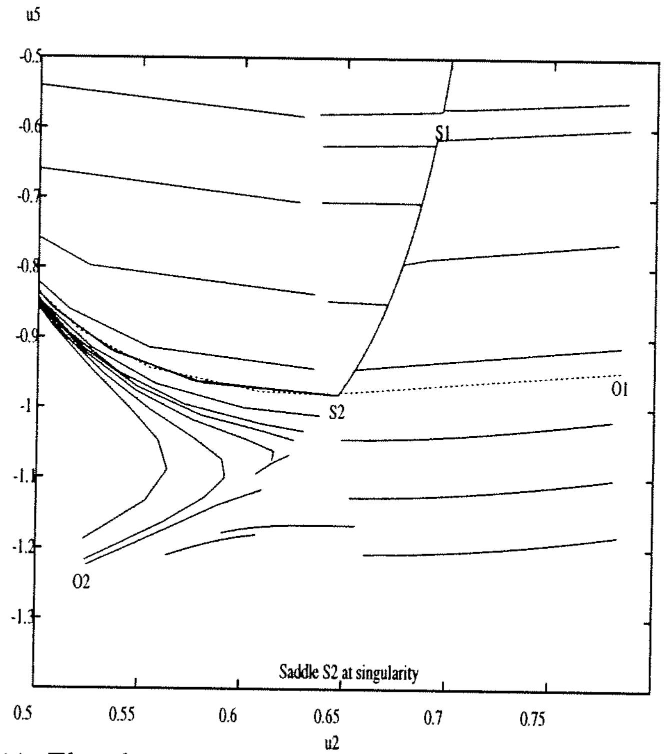

Fig.44. The phase portrait zoomed in around saddle S_2.

It is also interesting to note that by placing a node with two stable eigenvalues at the singularity changes one of the eigenvalues so that it is $-\infty$ on one side of the singularity and $+\infty$ on the other one. Therefore both a node and a saddle have the same eigenvalue structure when placed at the singularity. Another interesting property of this system is an invariant manifold $O - S_2 - N$ which touches the singularity manifold at saddle S_2.

5.8 A Cautionary Example

The last example in this section illustrates that care must be exercised when dealing with numerical issues in DAEs. Suppose that we consider the η only case of the MHD DAEs with $B^x = 0$, $s = u_2^l$ and the remaining components of the *left* state are chosen to satisfy

$$u_7^l + \frac{3}{4}[(u_5^l)^2 + (u_6^l)^2] - \frac{1}{2}su_1^l - \frac{1}{2}u_1^l[(u_3^l)^2 + (u_4^l)^2] = 0$$

Then we have the following DAE

$$u_5' = (u_2 - s)u_5 \tag{187a}$$

$$u_6' = (u_2 - s)u_6 \tag{187b}$$

$$0 = (u_2 - s)\left(u_5 + u_6 + \frac{3}{4}[u_5^2 + u_6^2]\right) \tag{187c}$$

The linearization around an equilibrium (u_5, u_6, u_2) is given by the pencil

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} u_2 - s & 0 & u_5 \\ 0 & u_2 - s & u_6 \\ (u_2 - s)(1 + \frac{3}{2}u_5) & (u_2 - s)(1 + \frac{3}{2}u_6) & u_5 + u_6 + \frac{3}{4}(u_5^2 + u_6^2) \end{bmatrix} \tag{188}$$

The surface given by the constraint (187c) consists of a cylinder C and a plane $P(u_2 = s)$ which is perpendicular to the cylinder. The equilibrium points are all of P and the line $u_5 = 0, u_6 = 0$ and u_2 arbitrary on C. We denote this line by L and the points on P but not on C by $P - C$.

However, this information does not correctly capture the full solution behavior of this example. To see this we examine the system more carefully. Let $(u_2 - s)Q$ be the righthand side of (187c). If $Q \neq 0$, that is $u \notin C$, then the DAE is semi-explicit index one and as noted earlier, conditions (A1)-(A4) in Definition 1.4 hold. However, it will be more convenient to consider (A1)-(A4) directly since we need the calculations later.

One differentiation of the DAE would suffice for considering solutions on P but two are needed for C. Differentiating the equations in (187) twice and forming the Jacobian we see that

$$[G_{y'}\ G_w] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & -1 & 0 & 0 & 0 & 0 \\ * & * & Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & * & -1 & 0 & 0 \\ 0 & * & * & 0 & * & * & 0 & -1 & 0 \\ * & * & * & * & * & Q & 0 & 0 & 0 \end{bmatrix} \tag{189}$$

where * is a possibly nonzero entry whose value is not important for this discussion. If $Q \neq 0$, then the rank of (189) is 8 (has corank 1). If only the first 6 rows are considered the matrix has rank 5 (has corank 1) and the matrix is still one full. One can easily show that (187) satisfies assuptions (A1)-(A4) and the requirements of theorem in Subsection 4.5 and (187) is solvable index one DAE on $P - C$.

At an equilibrium on $P - C$ we have the pencil

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & u_5 \\ 0 & 0 & u_6 \\ 0 & 0 & Q \end{bmatrix} \tag{190}$$

where Q is nonzero. The pencil is regular and does correctly capture the index and solution behavior.

Suppose, however, that we are on C so that $Q = 0$. Then the matrix (189) is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & -1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & * & -1 & 0 & 0 \\ 0 & * & * & 0 & * & * & 0 & -1 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 \end{bmatrix} \tag{191}$$

One-fullness and constant rank are invariant under invertible row operations that do not originate with the first three columns. Using the first two rows to zero below them and the 7th and 8th columns to zero to the left, we get

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \end{bmatrix} \tag{192}$$

which has rank no more than 7. But as noted earlier, the rank is 8 if $Q \neq 0$. A similar calculation, which we omit, works for 3 differentiations which is the maximum needed since the system has three variables. Thus (A3) is violated since the rank is not constant in a full neighborhood of any points on C and the DAE is not uniformly solvable on C or any submanifold of C.

We now look at the solutions on C. Since the problem appears to be index 2 we expect another constraint. Differentiating the constraint (187c), we get, after some simplification, that we have the additional constraint

$$u_5^2 + u_6^2 = 0 \tag{193}$$

on C. Thus $u_5 = 0$, $u_6 = 0$ which is the line L. We already know that L consists of a line of equilibria stretching up the side of the cylinder. However, this does not fully describe the solutions. If $u_5 = 0$ and $u_6 = 0$, then the DAE on C puts no restrictions on u_2. Thus the DAE on C has solutions $u_5 = 0$, $u_6 = 0$ and u_2 arbitrary. The DAE is not solvable. DAEs with an infinite number of solutions have been discussed elsewhere [17]. However, some sort of regularization, motivated by physical considerations, is needed to pick out a particular solution.

Finally, we note that the pencil (189) on C is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & u_5 \\ 0 & 0 & u_6 \\ 0 & 0 & 0 \end{bmatrix} \tag{194}$$

which is not regular.

parameter	equilibria: (x_2, x_5, x_6)	eigenvalues
p=17.50	(0.5000,1.0000,1.0000)	(0.4398,-0.0429)
	(0.7455,-0.2115,-0.2115)	(0.2026,0.1975)
	(0.5842,-1.0371,-1.0371)	(2.2231, 0.0413)
	(0.3200,0.1923,0.1923)	(-0.2117,-0.2228)
p=17.70	(0.5000,1.0000,1.0000)	(0.4192,-0.0480)
	(0.7462,-0.2424,-0.2424)	(0.1981,0.1981)
	(0.5943,-1.0370,-1.0370)	(3.671, 0.0463)
	(0.3253,0.2156,0.2156)	(-0.2089,-0.2227)
p=17.85	(0.5000,1.0000,1.0000)	(0.4044,-0.0518)
	(0.7466,-0.2661,-0.26661)	(0.1947,0.1866)
	(0.6018,-1.0360,-1.0360)	(7.6068,0.0500)
	(0.3292,0.2328,0.2328)	(-0.2066,-0.2226)
p=17.95	(0.5000,1.0000,1.0000)	(0.3949,-0.0543)
	(0.7274,-0.2822,-0.2822)	(0.1924,0.1832)
	(0.6068,-1.0350,-1.0350)	(33.5211,0.0525)
	(0.3318,0.2441,0.2441)	(-0.2050,-0.2225)
p=17.97	(0.5000,1.0000,1.0000)	(0.3930,-0.0548)
	(0.7468,-0.2855,-0.2855)	(0.1920,0.1826)
	(0.6079,-1.0347,-1.0347)	113.65,0.0530)
	(0.3323,0.2463,0.2463)	(-0.2047,-0.2225)
p=17.97829718	(0.5000,1.0000,1.0000)	(0.3923,-0.0550)
	(0.7468,-0.2869,-0.2869)	(0.1918,0.1823)
	(0.6082,-1.0346,-1.0346)	($\pm\infty$,0.0533)
	(0.3325,0.2473,0.2473)	(-0.2045,-0.2225)
p=18.00	(0.5000,1.0000,1.0000)	(0.3902,-0.0556)
	(0.7469,-0.2904,-0.2904)	(0.1913,0.1815)
	(0.6092,-1.0343,-1.0343)	(0.0537,-43.0570)
	(0.3331,0.2497,0.2497)	(-0.2042,-0.2225)
p=18.20	(0.5000,1.0000,1.0000)	(0.3721,-0.0604)
	(0.7471,-0.3238,-0.3238)	(0.1867,0.1744)
	(0.6191,-1.0306,-1.0306)	(0.0586,-3.9483)
	(0.3382,0.2719,0.2719)	(-0.2007,-0.2223)
p=18.50	(0.5000,1.0000,1.0000)	(0.3466,-0.0676)
	(0.7472,-0.3764,-0.3764)	(0.1795,0.1627)
	(0.6337,-1.0220,-1.0220)	(0.0661,-1.4918)
	(0.3458,0.3046,0.3046)	(-0.1952,-0.2218)
p=19.00	(0.5000,1.0000,1.0000)	(0.3078,-0.0789)
	(0.7456,-0.4738,-0.4738)	(0.1667,0.1388)
	(0.6582,-0.9962,-0.9962)	(0.0792,-0.5783)
	(0.3352,0.3575,0.3575)	(-0.2100,-0.2437)

Table 1: Equilibria and their eigenvalues during bifurcation.

6 Conclusions and Further Research Topics

6.1 Other Boundary Value and Boundary Control Problems for PDAEs

So far we have considered simple boundary value and boundary control problems for linear PDAEs. There are a number of other more complicated formulations, for example, a two point boundary value problem (TPBVP) for PDAEs which has not been analyzed in detail yet. One can expect to obtain several interesting results for a TPBVP for PDAEs. Perhaps some preliminary results are worth mentioning. First, one TPBVP for a simplified PDAE has the following form

$$\begin{aligned} Au_t + Bu_{xx} &= f(x,t) && \text{(195a)} \\ M_1u(0,t) = 0, &\quad M_2u(L,t) = 0 && \text{(195b)} \\ Q_1u(x,0) + Q_2(x,T) &= 0 && \text{(195c)} \end{aligned}$$

where Q_1 and Q_2 are some $r_1 \times n_1$ matrices (n_1 is the dimension of the regular part of pencil (A, B), r_1 is given below). The M_1 and M_2 denote boundary operators. A special case of (195c) includes the case when (195c) is replaced with $Q_1^*u(x,0) = 0$, $Q_2^*u(x,T) = 0$.

Problem (195) is known as a TPBVP for simplified linear PDAEs. A similar problem for linear DAEs was considered in [31], where it was proved that Q_1 and Q_2 must satisfy the following condition

$$r_1 \equiv rank[Q_1, Q_2] = core\text{-}rank(A_\omega) \tag{196}$$

where $core\text{-}rank(A_\omega) \equiv rank(A_\omega^D A_\omega)$, superscript D denotes the Drazin inverse [12] and $A_\omega = (\omega A - B)^{-1}A$. Condition (196) is not sufficient in case of (195). It may happen that the spectrum of the operator $\partial^2/\partial x^2$ intersects the spectrum of the modal TPBVP DAE (i.e. a TPBVP for a system of modal DAEs, see Subsections 3.4 and 4.1). This may yield nonunique solutions as the following example shows.

Example 6.1 *Consider the zero Neumann boundary conditions at $x = 0$ and $x = 1$ for system (195) with*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f(x,t) \equiv 0, \tag{197}$$

and the boundary conditions (195c) with

$$Q_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{198}$$

for $T = 1$.

Note that due to the zero Neumann boundary conditions $0 \notin \rho(P)$ ($\rho(P)$ is the resolvent set of $P \equiv \frac{\partial^2}{\partial^2 x}$). Also, since $0 \notin \rho(P)$, the nilpotent part will admit nonunique solutions. The regular part will also have nonunique solutions, but the reason for this is quite different. It is easily seen that the TPBVP DAE has nonunique solutions $u(x,t) = (c, 0, \alpha(t))'$, where $c \in \mathbf{R}$, and $\alpha(t)$ is a continuous function such that $\alpha(1) = 0$. The point spectrum of the spatial operator is $0, (2k+1)\frac{\pi}{2}$; k=0,1,2,... . The first eigenvalue in this set is responsible for nonuniqueness of the solution for the nilpotent part. Eigenvalues $\frac{(2k+1)\pi}{2}$ for k=0,2,4,..., are responsible for nonuniqueness of the solution of the regular part. Eigenvalues with k=1,3,5,... do not cause any problems in this example. The analysis is as follows. The regular part of the DAE is

$$\dot{u}_{k,reg}(t) - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (2k+1)\frac{\pi}{2} u_{k,reg}(t) = 0. \tag{199}$$

The first two rows of Q_1 and Q_2 together with the BVP for the regular part yield for $u_{k,reg}(t) = (u^1_{k.reg}(t), u^2_{k,reg}(t))^T$

$$\left[\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} exp\Big(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (2k+1)\frac{\pi}{2} \Big) \right] u_{k,reg}(0) = 0, \tag{200}$$

which gives

$$\begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} u_{k,reg}(0) = 0, \quad k = 1, 3, 5, ..., \tag{201}$$

and

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} u_{k,reg}(0) = 0, \quad k = 0, 2, 4, ... \tag{202}$$

Therefore the even elements of the point spectrum of the spatial operator allow $u^1_{k,reg}(0) \neq 0$. This yields $u_1(x,t) = c \in \mathbf{R}$, $u_2(x,t) = 0$, and $u_3(t) = \alpha(t)$, $u_3(T) = 0$.

The eigenvalue 0 in the point spectrum of the spatial operator is responsible for nonuniqueness of $u_3(x,t)$, while the spectrum of the regular part of the DAE BVP intersects even eigenvalues $\frac{(2k+1)\pi}{2}$, $k = 0, 2, 4, ...$, yielding nonuniqeness of the solution for the regular part. Note that $f(x,t) \equiv 0$ above.

There may also be another type of nonuniqueness associated with the zero eigenvalue of the spatial operator if $f_k(t)$ is such that it is nonzero at $t = 0$ and/or $t = T$, and if there exists coupling between reqular and nilpotent subsystems. This happens when the solution of the nilpotent part is nonunique and there exists a coupling of boundary conditions at $t = 0$ or $t = T$ due to (198). In other words the nonuniqueness of $u_{k,nil}(t)$ is transmitted onto nonuniqueness of $u_{k,reg}(t)$. The above is easily seen if one considers the formula for the general solution of the BVP for (195) given by

$$\begin{aligned} u_{k,reg}(t) &= e^{-J\lambda_k t}[Q_1^{n_1} + Q_2^{n_2} e^{-J\lambda_k T}]^{-1}\Big(\sum_{i=1}^{n_0}(-1)^{i+1}\lambda_k^{-i}[Q_1^{n_2} J_0^{i-1} f_{k,nil}^{(i-1)}(0) \\ &\quad + Q_2^{n_2} J_0^{i-1} f_{k,nil}^{(i-1)}(T)] + Q_1^{n_1}\int_0^t e^{J\lambda_k s} f_{k,reg}(s)ds \\ &\quad - Q_2^{n_1} e^{-J\lambda_k T}\int_t^T e^{J\lambda_k s} f_{k,reg}(s)ds\Big) \end{aligned} \tag{203a}$$

where $Q_j^{n_l}$, $j = 1,2$; $l = 1,2$, are matrices defining the BVP (after transforming the PDAE into cannonical form) and

$$Q_1 \equiv [Q_1^{n_1}, Q_1^{n_2}], \qquad Q_2 \equiv [Q_2^{n_1}, Q_2^{n_2}] \tag{204}$$

where n_1 and n_2 are dimensions of the regular and nilpotent parts, respectively. Particular forms of $Q_j^{n_l}$ may give nonuniqueness of $u_{k,reg}(t)$ if $f_k(0)$ and/or $f_k(T)$ are nonzero.

The above example illustrates just one of many possibilities in boundary value or boundary control problems that may be of interest when dealing with PDAEs.

6.2 Generalization of the Singularity Induced Bifurcation in DAEs

Since the travelling wave solutions in dissipative systems of conservation laws are, under some mild conditions, equivalent to the shock solutions in nondissipative systems, it seems interesting to examine how the singularity induced bifurcation impacts shock structure. No such analysis exists so far.

Also, there are other areas of applications in which travelling waves play an important role. There is a system of 5 nonlinear PDEs in hypoplasticity for which the Riemann problem has been recently analysed in [37].

Another aspect of the singularity induced bifurcation is also worth analysing. Namely, one may be interested in generalization of the singularity induced bifurcation theorem for multi-parameter bifurcation or the case when $D_v g(u,v)$ has a multiple zero eigenvalue. The following simple example has double zero eigenvalue of $D_v g(u,v)$ at the singularity.

Example 6.2 *Let*

$$\begin{aligned} u_1' &= p - v_1 - v_2 && (205a) \\ u_2' &= p - v_1 - v_2^2 && (205b) \\ 0 &= u_1 - v_1^2 + v_1 v_2 && (205c) \\ 0 &= u_2 - v_1^2 && (205d) \end{aligned}$$

Since

$$det\frac{\partial g}{\partial v} = det\begin{bmatrix} -2v_1 & v_1 \\ -2v_1 & 0 \end{bmatrix} = 2v_1^2, \tag{206}$$

the DAE is singular if $v_1 = 0$.

It is easy to check that the following two equilibria of this system are placed at the singularity

$$(u_1^0, u_2^0, v_1^0, v_2^0, p^0) = \begin{cases} (0,0,0,0,0) \\ (0,0,0,1,1) \end{cases}$$

Note that $\frac{\partial g}{\partial v}$ *has a double root at each of these equilibria and*

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\left[\frac{\partial g}{\partial v}\right]^{-1}\frac{\partial g}{\partial u} = \begin{bmatrix} \frac{1}{v_1} & -\frac{3}{2v_1} \\ \frac{2v_2}{v_1} & -\frac{1+4v_2}{2v_1} \end{bmatrix} \tag{207}$$

has two eigenvalues diverging through $\pm\infty$ *at both equailibria, but the pattern of divergence is different at each equilibrium.*

6.3 MHD DAE as a Test Problem for a General Purpose Numerical DAE Solver

The MHD DAE presented in some detail in Subsections 5.5-5.7 seems to be an ideal test problem for any general purpose DAE solver. There exists no such solver today and several existing DAE packages are applicable only in special cases of (1). Those special cases include for example only index 1 DAE, a DAE in Hessenberg form, or a DAE with time-dependent Jacobians only. Note that as shown in Subsections 5.5, 5.6 the MHD DAE may not have any of the above properties. The MHD DAE may have Jacobians dependent not only on t, but also on x and x' (Subsection 5.6.2). Note that the general MHD DAE includes many special cases depending either on the dissipative mechanism (coefficients η, κ, μ, ν) and on the type of process (e.g. *adiabatic* with constant pressure, or *isotermal* with constant temperature). This yields different substructures of (130), some with singularities and/or impasse points some without singularities and without impasse points. Note also that the singularity induced bifurcation and our desire to connect two equilibria on two different sides of singularity via an intermediate equilibrium at the singularity will be a particularly difficult problem to handle by a DAE solver. This is because one is not interested in some neighborhood of the equilibrium at the singularity but exactly at that one single point (equilibrium). Any DAE solver would have to exactly hit the intermediate equilibrium. This presents additional difficulties in any predictor-corrector method. The ideal goal would be of course to treat the MHD DAE (130) as a *black box* and develop a general purpose DAE solver which can handle all particular cases examined in Subsections 5.6 and 5.7 no matter what the dissipative mechanism is, no matter whether we have singularities and/or impasse points, and no matter what the nature of the physical process is. It seems that the above issues define a long range research project in DAEs but work in this direction is worth being undertaken.

6.4 Other Systems of Conservation Laws

Although the systems of gas dynamics and MHD conservative laws have been analyzed in some detal in the preceeding sections, similar analysis can be carried out for other systems described by (102). It is well known that many nontrivial mathematical problems occur for the conservation laws in elastoplasticity [35]. The 7 nonlinear non-dissipative equations in elasticity have the following form

$$\begin{aligned} \rho_t + (\rho v^1)_x &= 0 & & \quad (208a) \\ (\rho f^i)_t + (rhof^i v^1)_x - (v^i)_x &= 0 \quad i = 2,3 & & \quad (208b) \\ (\rho v^i)_t + (\rho v^i v^1)_x - (\sigma^i)_x &= 0 \quad i = 1,2,3 & & \quad (208c) \\ S_t + v^1 S_x &= 0 & & \quad (208d) \end{aligned}$$

This system of 7 equations has the general form (102) with $u = [f^1, f^2, f^3]$ being a deformation gradient vector, $\sigma(f, S) = [\sigma^1, \sigma^2, \sigma^2]$ is the stress vector for a uniaxial deformation, $v = [v^1, v^2, v^3]$ is the velocity vector, S is entropy, and $\rho^{-1} = f^1$. The Jacobian of $F(u)$ in (102) for the above system is

$$F'(u) = \begin{bmatrix} v^1 & & & -\rho^{-1} & & & \\ & v^1 & & & -\rho^{-1} & & \\ & & v^1 & & & -\rho^{-1} & \\ -\rho^{-1}\sigma^1_{f^1} & -\rho^{-1}\sigma^1_{f^2} & -\rho^{-1}\sigma^1_{f^3} & v^1 & & & -\rho^{-1}\sigma^1_S \\ -\rho^{-1}\sigma^2_{f^1} & -\rho^{-1}\sigma^2_{f^2} & -\rho^{-1}\sigma^2_{f^3} & v^1 & & & -\rho^{-1}\sigma^2_S \\ -\rho^{-1}\sigma^3_{f^1} & -\rho^{-1}\sigma^3_{f^2} & -\rho^{-1}\sigma^3_{f^3} & v^1 & & & -\rho^{-1}\sigma^3_S \\ & & & & & & v^1 \end{bmatrix} \quad (209)$$

where σ^i_m denotes partial derivative of σ^i w.r.t. m.

The characteristic speeds can be found as follows

$$det(\lambda I - F'(u)) = \rho^{-6}(\lambda - v^1) det[\rho^2(\lambda - v^1)^2 I - C] = 0 \quad (210)$$

where $C = (\sigma^i_{f^j}), i, j = 1, 2, 3$. One speed is $\lambda = v^1$. Also, if $c = \rho^2(\lambda - v^1)^2$ (an eigenvalue of C), then for each c we have two characteristic speeds: $\lambda = V^1 \pm \rho^{-1}c^{1/2}$, and λ's are real if C is positive definite. That is the system is thermodynamically stable. In this case we have a total of 7 different real characteristic speeds and the system is *genuinely* nonlinear. This yields equivalence between the shock solutions of the hyperbolic systems in elasticity and the traveling solutions in the corresponding dissipative sustem. One may expect to obtain different types of DAEs in elasticity, depending on the dissipative terms involved.

6.5 DAEs and Catastrophe Theory

As was shown in Section 5 the MHD DAEs can have constraint manifolds of different shapes. The equation describing the constraint surface is a quadratic polynomial in 3 variables. Since there are many parameters involved in the original system (130), the shape of this 3D surface may be quite complicated. One may expect that other systems of conservation laws may have constraint manifolds of degree higher than 2. If the degree is at least 3, then various interesting shapes may be involved. This seems to show that there are obvious links between DAEs and catastrophy theory. At present, those links are not well understood and when analyzed may yield interesting results. The catastrophy theory examines various algebraic equations in the 3D case such as, for example, $y^2 = zx^2$ (Whitney umbrella), or $y^2 = z^3x^2$ (folded Whitney umbrella). The other type of shapes are: *swallowtail*, *pedal loci*, *cusp*, and *Legendre singularities* [41], [42].

6.6 Conclusions

Analysis of DAEs is an important topic in the current research in applied and numerical mathematics. As yet, there are many open problems and even some elementary questions have not been answered. One particular area of research of this kind is the broad area of partial differential equations. It is natural to expect that with the advancement in computer technology and the development of sophisticated methods of mathematical modeling, we will face more and more complicated models of real systems to be analysed. Mixed systems of ordinary and partial differential equations are already used to model many dynamical systems in mechanical, electrical and chemical engineering [25], [26], [49].

Our goal in this thesis has been to analyse such systems. We have defined three different indices for PDAEs, compared them with each other and with the indices in finite dimensional DAEs. It has been shown that even in linear case, the indices of PDAEs can differ from each other. This is not the case in linear finite dimensional DAEs. Different type of spatial approximation may also yield DAEs with different indeces even when for the same PDAE. This has been illustrated by approximation with the finite difference and finite element MOL. Additionally, the index of a PDAE may depend on the weighting coefficients, as is shown by an example of the boundary control problem in Section 4. Generally speaking, the PDAE exhibit much richer behavior than finite dimensional DAEs and one needs to be cautious with the application of different numerical integrators when dealing with PDAEs.

Another area where DAEs naturally occur is nonlinear PDEs. It has been shown that certain types of solutions (e.g. traveling waves) may lead to DAEs. We have considered systems of conservation laws in gas dynamics and MHD and derived MHD DAEs in both semi-explicit and conservation forms (Section 5). It has been shown that some well-known notions in systems of conservation laws can be linked to corresponding notions in DAEs.

One such example is the correspondence between the *sonic* points and the *higher index* DAE.

We have also shown that the MHD system may be subjected to singularity induced bifurcations. In some cases, by placing an equilibrium at the singularity, one is able to find traveling waves between equilibria lying on the opposite sides of that singularity. The equilibrium at the singularity serves as an intermediate point. In some cases this equilibrium is reached/left in finite time. This type of traveling wave solution may lead in the future to the proof of existence of some type of new shock solutions in non-dissipative MHD systems.

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7 Appendix I

In the proof below only the most important responses from MAPLE are printed out, i.e. formulas for $u1$, $u2$, $u3$, $u4$, $u5$ and $u7$ as functions of u6. Some other intermediate formulas are very long. There is nothing special about the fact that $u1$, $u2$, $u3$, $u4$, $u5$ and $u7$ are function of $u6$. One can choose another variable and express other variables as function of that particular one.

In what follows we do all calculations symbolically for the general 7 numbers $ul1, ..., ul7$ ('left' state), wave speed s, and constant parameters gam (γ), Bx (B^x). Compute first the 'left' value of $Pstar$ ($= P^*$). Note that p (static pressure) is eliminated from the two equations given below (127):

$Pstarl := (gam - 1) * (ul7 - (Bx^2 + ul5^2 + ul6^2)/2 - ul1 * (ul2^2 + ul3^2 + ul4^2)/2) + (Bx^2 + ul5^2 + ul6^2)/2$:

The general expression for $Pstar$ in terms of the seven semistate variables $u1, ..., u7$ is:

$Pstar := (gam - 1) * (u7 - (Bx^2 + u5^2 + u6^2)/2 - u1 * (u2^2 + u3^2 + u4^2)/2) + (Bx * Bx + u5 * u5 + u6 * u6)/2$:

Solve the first equation in (130) for $u1$:

$u1 := solve(-s * u1 + s * ul1 + u1 * u2 - ul1 * ul2 = 0, u1)$:

Substitute $u1$ into the third equation in (130) with the zero right hand side:

$u3 := solve(simplify(-s * u1 * u3 + s * ul1 * ul3 + u1 * u2 * u3 - Bx * u5 - ul1 * ul2 * ul3 + Bx * ul5) = 0, u3)$;

$$u3 := -\frac{s\, ul1\, ul3 - Bx\, u5 - ul1\, ul2\, ul3 + Bx\, ul5}{-s\, ul1 + ul1\, ul2}$$

Substitute $u1$ into the fourth equation in (130) with the zero right hand side:

$u4 := solve(simplify(-s * u1 * u4 + s * ul1 * ul4 + u1 * u2 * u4 - Bx * u6 - ul1 * ul2 * ul4 + Bx * ul6) = 0, u4)$;

$$u4 := -\frac{s\, ul1\, ul4 - Bx\, u6 - ul1\, ul2\, ul4 + Bx\, ul6}{-s\, ul1 + ul1\, ul2}$$

Substitute $u1$, $u3$ and $u4$ into formula for $Pstar$ (see above):

$Pstar := simplify((gam - 1) * (u7 - (Bx^2 + u5^2 + u6^2)/2 - u1 * (u2^2 + u3^2 + u4^2)/2) + (Bx * Bx + u5 * u5 + u6 * u6)/2)$:

Use $Pstar$ given above and solve the second equation in (130) (with the zero right hand side) for $u7$:

$$u7 := solve(-s*u1*u2+s*ul1*ul2+u1*u2*u2+Pstar-ul1*ul2*ul2-Pstarl = 0, u7):$$

Use $u3$ computed above and solve the fifth equation in (130) (with the zero right hand side) for $u2$:

$$u2 := simplify(solve((-s*(u5-ul5)+u5*u2-Bx*u3-(ul2*ul5-Bx*ul3)) = 0, u2)):$$

Use $u2$ computed above and solve the sixth equation in (130) (with the zero right hand side) for $u5$:

$$u5 := simplify(solve((-s*(u6-ul6)+u6*u2-Bx*u4-(ul2*ul6-Bx*ul4)) = 0, u5));$$

$$u5 := \frac{u6\ ul5}{ul6}$$

Use $u5$ to simplify $u2$:

$$u2 := simplify(u2);$$

$$u2 := (-u6\ s^2\ ul1 + u6\ s\ ul1\ ul2 + s^2\ ul1\ ul6 - 2\,s\ ul1\ ul2\ ul6 + u6\ Bx^2 - Bx^2\ ul6 + ul2^2\ ul6\ ul1) \Big/ (\,ul1\,(-s + ul2\,)\,u6\,)$$

Simplify $u3$ so that now $u3$ is a function of $u6$:

$$u3 := simplify(u3);$$

$$u3 := \frac{-s\ ul1\ ul3\ ul6 + Bx\ u6\ ul5 + ul1\ ul2\ ul3\ ul6 - Bx\ ul5\ ul6}{(-s + ul2\,)\ ul6\ ul1}$$

Simplify $u1$ so that now $u1$ is a function of $u6$:

$$u1 := simplify(u1);$$

$$u1 := \frac{ul1^2\,(-s + ul2\,)^2\ u6}{s^2\ ul1\ ul6 - 2\,s\ ul1\ ul2\ ul6 + u6\ Bx^2 - Bx^2\ ul6 + ul2^2\ ul6\ ul1}$$

Simplify Pstar so that now $Pstar$ is a function of $u6$:

$$Pstar := simplify(Pstar):$$

Simplify $u7$ so that now $u7$ is a function of $u6$:

$u7 := simplify(u7)$:

Since $u1$, $u2$, $u3$, $u4$, $u5$ and $u7$ are given above (all are functions of $u6$), therefore we can use the seventh equation in (130) (with the zero right hand side) and simplify it so that its left hand side will be a polynomial in $u6$ only, say $poly(u6)$. Then solving the equation $poly(u6) = 0$ one gets several equilibria for $u6$. The corresponding values for $u1$, $u2$, $u3$, $u4$, $u5$ and $u7$ can be easily found by using the formulas derived above. The $poly(u6)$ is of 4 degree as the following calculation shows:

$f(u6) := simplify(-s * u7 + s * ul7 + (u7 + Pstar) * u2 - Bx * (Bx * u2 + u3 * u5 + u4 * u6) - (ul7 + Pstarl) * ul2 + Bx * (Bx * ul2 + ul3 * ul5 + ul4 * ul6))$:

Check if $ul6$ is a zero of $f(u6)$:

$subs(u6 = ul6, f(u6))$;

$$0$$

Finally, $poly(u6)$ =numerator of $f(u)$:

$poly(u6) := simplify(f(u6) * (ul2 - s) * ul1 * ul6^2 * u6^2 * (gam - 1) * (-2))$:

Since $deg[poly(u6)] = 4$ therefore we can have at most 4 equilibria of the travelling wave DAEs (130). Note that one of those equilibria should be the assumed 'left' equilibrium, which means that $poly(ul6)$ should be zero. This is in fact the case as the following substitution shows. If the 'left' equilibrium is hyperbolic then one can assure the existence of at least one more equilibrium.

$subs(u6 = ul6, poly(u6))$;

$$0$$

Note that $ul6$ is really a root of $poly(u6) = 0$. Additional analysis shows that if $ul6 \neq 0$, then $ul6$ is a single root. Close examination of the 4th degree polynomial poly(u6) shows that its leading coefficient is nonnegative and equal $Bx^2(ul6^2 + ul5^2)$. Also the constant coefficient of this polynomial can be written in a closed form $ul6^4(gam+1)[(s-ul2)^2ul1 - Bx^2]^2$, i.e. it is nonnegative. The above facts are equivalent to the fact that there exists at least one more root (except $ul6$) of $poly(u6)$. In addition, this additional root has the same sign as $ul6$ (or equals zero). This is because the constant and leading coefficients are both nonnegative. Therefore the MHD travelling wave DAE has at most 4 and at least 2 equilibria.